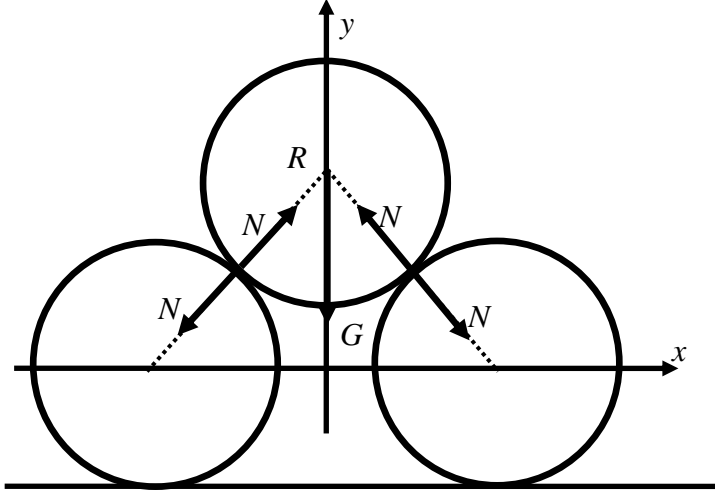# I. Spheres in Contact

a). $mgh = mgy + 2\dfrac{mv_x^2}{2} + \dfrac{mv_y^2}{2}$, $h = \sqrt{3}R$,   (1)   **0.75p**

$x^2 + y^2 = 4R^2$.   (2)   **0.75p**

From Eq. (2):

$x\dot{x} + y\dot{y} = 0 \Leftrightarrow xv_x + yv_y = 0$,   (3)   **1p**

$x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = 0 \Leftrightarrow x\ddot{x} + y\ddot{y} + v_x^2 + v_y^2 = 0$.   (4)   **1p**

From Eqs. (1-3):

$v_x^2 = \dfrac{2g(h-y)y^2}{4R^2 + y^2}$,   (5)   **0.5p**

$v_y^2 = v_x^2\left(\dfrac{y}{x}\right)^2 = \dfrac{2g(h-y)(4R^2 - y^2)}{4R^2 + y^2}$.   (6)   **0.5p**

$-mg + 2N\cos\theta = m\ddot{y}$,   (7)   **0.75p**

$N\sin\theta = m\ddot{x}$.   (8)   **0.75p**

From Eqs. (4-6, 7, 8):

$N = mg\dfrac{-8\sqrt{3} + 12y/R + (y/R)^3}{4\left[4 + (y/R)^2\right]} \Rightarrow$

Reaction N cancels for $y_0 = x_0 R$.   (9)   **0.5p**

One introduces y_0 inEqs. (5, 6) and one obtains,:

$v_x^2(y = y_0) = \dfrac{2gRx_0^2(\sqrt{3} - x_0)}{4 + x_0^2}$,

$$v_y^2(y = y_0) = \frac{2gRx_0^2(\sqrt{3} - x_0)(4 - x_0^2)}{4 + x_0^2}.$$ (10)   **0.5p**

b)

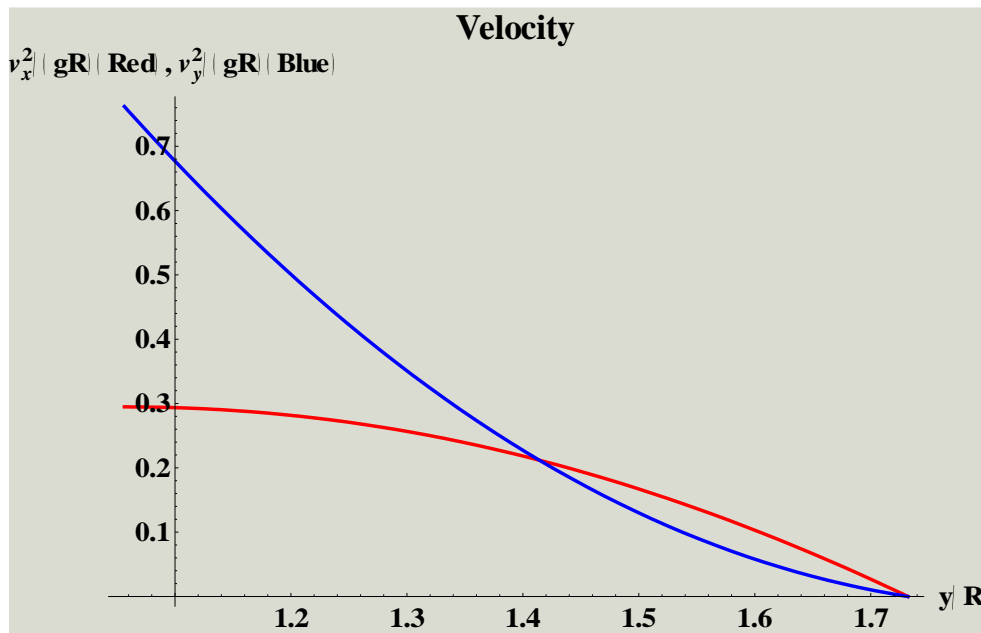$$v_3 \equiv v_x; \quad \frac{dv_x^2}{dy} = \frac{-2gy\left(-8\sqrt{3}R^3 + 12R^2y + y^3\right)}{\left(4R^2 + y^2\right)^2} \quad \text{is negativ for } \left(x_0, \sqrt{3}\right)R, \text{ so } v_x^2$$

indreases with time. (11)   **0.75p**

$$v_y^2 = v_x^2\left(\frac{x}{y}\right)^2 \text{ and } \frac{x}{y} \text{ indreases with time} =>$$

v_y^2 indreases with time. (12)   **0.75p**

Graphs for $v_x^2/(gR)$, $v_y^2/(gR)$ for $y/R \in [x_0, \sqrt{3}]$:

Velocity

$v_x^2 \, (\, gR \,)$ (**Red**), $v_y^2 \, (\, gR \,)$ (**Blue**)

Scheme for the graph $v_x^2(y)$ **0.75p**

c) The maximum height for sphere 1 after collision with the table:

$$h_{max} = h - \frac{v_x^2(y = y_0)}{g}$$

$$= R\frac{4\sqrt{3} - \sqrt{3}x_0^2 + 2x_0^3}{4 + x_0^2} \approx 1.43728 \, R$$ (13)   **0.75p**

II.   The apparent air temperature in a windy winter day

Consider Ox the direction perpendicular to the skin.

The heat conduction equation (Fourier's law): $q(x) = -\kappa \dfrac{\partial T}{\partial x}$(1)   **1p**

For distance $\gg$ mean free path, $q(x) = q = \text{const.}$and the temperature of the surroundings $T(d) = T_{surr}$, the solution is: $T(x) = -\dfrac{q}{\kappa}(x - d) + T_{surr}$(2)   **1p**

The Newton law: $q = -h(T_{ext} - T_{body})$(3)   **2p**

where$T_{ext} = T(0)$is the temperature of the surroundings in the vicinity of the interface.
From eqs. (2) and (3) one obtains $T_{ext} = \dfrac{hd}{\kappa}(T_{body} - T_{ext}) + T_{surr}$(4)   **1p**

Eq. (4) is valid in the conditions of calm atmosphere.
Turbulence: The wind produces a strong convection, $T_{ext} = T_{surr}$and the Newton law
is $q' = -h(T_{surr} - T_{body})$(5)   **1p**

In the case of a calm atmosphere, $T(d) = T'_{surr}$ and: $T(x) = -\dfrac{q'}{\kappa}(x - d) + T'_{surr}$ **1p**

At the body interface $T(0) = T_{surr} = \dfrac{q'd}{\kappa} + T'_{surr}$(6)

The equivalent of eq. (4) for the calm atmosphere and $T(0) = T_{surr}$,
$$T_{surr} = \dfrac{hd}{\kappa}(T_{body} - T_{surr}) + T'_{surr}$$(7)

Finally $T'_{surr} = T_{surr} - (T_{ext} - T_{surr})\dfrac{T_{body} - T_{surr}}{T_{body} - T_{ext}}$ **1p**

With the numerical values $T_{ext} = 5^\circ C$; $T_{body} = 37^\circ C$and $T_{surr} = -5^\circ C$ one
obtains $T'_{surr} = -18^\circ C$ **1p**

III. Huygens cycloidal pendullum

1. We have

$$\dot{x}_1 = a(1+\cos\alpha)\dot{\alpha} \quad ; \quad \dot{x}_2 = a\sin\alpha\,\dot{\alpha}$$ **0.5 p**

The kinetic energy is:

$$E_{kin} = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) = ma^2(1+\cos\alpha)\dot{\alpha}^2 = 2ma^2\cos^2\left(\frac{\alpha}{2}\right)\dot{\alpha}^2$$ **0.5 p**

The potential energy is:

$$U = mga(1-\cos\alpha) = 2mga\sin^2\left(\frac{\alpha}{2}\right)$$ **1 p**

The expression of the Lagrange function is:

$$L = 2ma^2\cos^2\left(\frac{\alpha}{2}\right)\dot{\alpha}^2 - 2mga\sin^2\left(\frac{\alpha}{2}\right)$$ **1 p**

2. We change the variable as follows

$$\beta := 2\sin\left(\frac{\alpha}{2}\right)$$

The Lagrange function becomes

$$L = 2ma^2\dot{\beta}^2 - \frac{1}{2}mga\beta^2$$

With the notations: $m' = 4\,m\,a^2$ and $k = m\,g\,a$, the Lagrange function reads

$$L = \frac{1}{2}m'\dot{\beta}^2 - \frac{1}{2}k\beta^2\,,$$

which is identical to the Lagrange function of the linear harmonic oscillator **1 p**

The Lagrange equation is

$$\ddot{\beta} + \frac{g}{4a}\beta = 0\,.$$

The solution of this differential equation is

$$\beta(t) = A\cos\left(\frac{1}{2}\sqrt{\frac{g}{a}}t\right) + B\sin\left(\frac{1}{2}\sqrt{\frac{g}{a}}t\right)$$ **1 p**

We need to determine the coefficients A and B. By using the initial condition
$x_2(t=0) = 2\,a = a[1-\cos\alpha(0)]$, we obtain
$\cos\alpha(0) = -1$, i.e. $\alpha(0) = \pm\pi$.
Further we use the other initial condition
$x_1(t=0) = -a\pi = a\,[\alpha(0) + \sin\alpha(0)]$, which leads to $\alpha(0) = -\pi$.
This is equivalent to $\beta(0) = -2 = A$ **1 p**

Also we have $\dot{\beta}(0) = 0$. The derivative of β with respect to time is

$$\dot{\beta}(t) = -\frac{1}{2}\sqrt{\frac{g}{a}}A\sin\left(\frac{1}{2}\sqrt{\frac{g}{a}}t\right) + \frac{1}{2}\sqrt{\frac{g}{a}}B\cos\left(\frac{1}{2}\sqrt{\frac{g}{a}}t\right),$$

which leads to B = 0. Therefore, the law of motion is:

$$\beta(t) = -2\cos\left(\frac{1}{2}\sqrt{\frac{g}{a}}\,t\right)$$

1 p

3. Since β is a periodic function, the period of oscillation is obtained as follows

$\frac{1}{2}\sqrt{\frac{g}{a}}\,T = 2\pi$, which is equivalent to

$$T = 4\pi\sqrt{\frac{a}{g}}$$

2 p

IV. Piezoelectric voltage from super-currents

1. No matter the turns are made of normal or superconducting material, *all wires will experience the same electromagnetic force when under the action of an external magnetic field*. With parallel super-currents flowing, all turns will experience *attracting forces* (**0.5 points**) and *will compress the piezo stacked ceramic* (**0.5 points**). The turns are now separated by $d + \Delta d$ and a voltage drop $U = \gamma N \Delta d$ develops across the stacked transducer. At equilibrium, the resulting compression force from super-currents is balanced by the Hooke's elastic force in piezoceramic. (**1 point**).

NOTE: It might possible that some students will argue that turns are not driven by electromagnetic forces due to the expulsion of magnetic field from superconductors (Meissner-Ochsenfeld efect). However, *the argument is not valid* since the electrical currents in superconductors are sustained by the surface region (as shown by Ampere's law) penetrated by the magnetic field.

2. When reversing the flow of supercurrents we still have parallel currents, and the attracting character of electromagnetic forces is preserved. The piezoelectric voltage exhibits even dependence on I. (**1 point**)

3. As a result of streching or compressing the stack, the magnetic inductance slighty changes. (**0.5 points**) If the super-current will stay the same, a flux variation $\Delta \Phi$ will be produced with the result of an induced electromotive force (Fraday's law). (**1 point**) In a superconductor, such an induced e.m.f. will lead to infinite acceleration of charges. Hence, to preserve the magnetic flux, the supercurrents need to obey the constraint (**1.5 points**)

$$\Phi = L_1 I_1 = L_2 I_2 = const . \tag{1}$$

4. The key point is to analize *how the super-current depends on the length of the stacking transducer*. For two different currents, the inductances are given by

$$L_1 = \mu_0 \frac{N^2 S}{(N-1)d_1} \ , \ L_2 = \mu_0 \frac{N^2 S}{(N-1)d_2} \ , \qquad \text{(\textbf{0.5 points}) (2a)}$$

where μ_0 is the magnetic permeability, $S = \pi R^2$ the area, $d_{1,2}$ the separation of turns with the super-currents $I_{1,2}$ flowing. Then, (1) yields

$$I_2 = I_1 d_2 / d_1 . \qquad \text{(\textbf{0.5 points}) (2b)}$$

A switching between two different supercurrents $I_1 \to I_2$, will change in the magnetic energy stored in transducer

$$\Delta W_{1 \to 2} = \frac{L_2 I_2^2}{2} - \frac{L_1 I_1^2}{2} = \frac{\mu_0 N^2 S I_1^2}{2(N-1)d_1^2} \Delta d_2 . \qquad \text{(\textbf{1 point}) (3a)}$$

As a consequence, *an amount of work is done* for any change between two super-currents. From (3), we deduce that at any given super-current

$$W_{0 \to I} = \frac{\mu_0 N^2 S I^2}{2(N-1)d^2} \Delta d . \qquad \text{(\textbf{1 point}) (3b)}$$

At equilibrium, the resulted compression from magnetic forces is balanced by the elastic Hooke's force $k(N-1)\Delta d$

$$F(I) = \frac{\partial W_{0 \to I}}{\partial N \Delta d} = \frac{\mu_0 N S I^2}{2(N-1)d^2} = k(N-1)\Delta d \quad . \qquad \textbf{(0.5 points)} \text{ (4a)}$$

At any given super-current I flowing, the compression of the stack is

$$\Delta d = \frac{\mu_0 N S I^2}{2k(N-1)^2 d^2} \ , \qquad\qquad\qquad \text{(4b)}$$

This compression generates a piezoelectric voltage drop

$$U(I) = \gamma N \Delta d = \frac{\mu_0 \gamma N^2 \pi R^2 I^2}{2k(N-1)^2 d^2} \qquad \textbf{(0.5 points)} \quad \text{(5)}$$

V.   Sum Rules

1. Show that $m_k = \langle E_0 | F(H - E_0)^k F | E_0 \rangle$ **2 p**

2. The proof of $m_1 = \frac{1}{2} \langle E_0 | [F, [H, F]] | E_0 \rangle$ and the answer that m_1 represents the total
energy transferred to the system by the excitation F **2 p**

3. The explicit proof of $m_1 = \frac{\hbar^2}{2m} \langle E_0 | \sum_{i=1}^{A} \left(\nabla f(\vec{Q}_i) \right)^2 | E_0 \rangle$ by taking into account
the fundamental commutation relations   **1 p**

4. The term $\frac{\hbar^2}{2m} \langle E_0 | \sum_{i=1}^{A} \left(\nabla f(\vec{Q}_i) \right)^2 | E_0 \rangle$ is related to the energy absorbedby all
nucleons and therefore the energy weighted sum rule encodes the energy
 conservation   **1 p**

5. $\sigma_D = \frac{2\pi^2 e^2 \hbar}{mc} \frac{NZ}{A}$ **3 p**

VI. Dynamical Symetries

1. The Runge-Lenz operator has to be Hermitian **4 p**

2. Indicate the condition to be a constant of motion: $\left[H, \vec{M} \right] = 0$ **1 p**

3. The explicit proof of $\left[H, \vec{M} \right] = 0$ **1 p**

4. Show that $\vec{M}$ behaves like a vector operator i.e. $\left[L_i, M_j \right] = i\hbar \varepsilon_{ijk} M_k$ **2 p**

4. Explicit proof of $\vec{M} \cdot \vec{L} = 0$ **1 p**

VII. Neutron Star

(a) Expression of the nuclear density is the following:

$$\rho = \frac{M(A,Z)}{v} = 3\frac{M(A,Z)}{4\pi R^3}$$

If a spherical body is built, layer with layer, a radius r can be obtained, as well as a new mass $M_0(A,z)r^3/R^3$. An additional new layer having the dimension dr involves an increase in mass given by $4\pi r^2\rho dr$. Taking into account that this layer is attached at a system having a radius r, the gravitational potential has the following form:

$$-g\frac{M(A,Z)r^2}{R^3}$$

Therefore, the contribution of the gravitational potential to the energy has the following form:

$$dU = -\frac{gM(A,Z)r^3}{R^3}4\pi R^3\rho dr = -g\frac{M(A,Z)}{R^3}4\pi\frac{3M(A,Z)}{4\pi R^3}r^4 dr =$$

$$= -3g\frac{M^2(A,Z)}{R^6}r^4 dr$$

3 p

The potential energy for whole system can be obtained integrating the previous relation, namely:

$$U = \int_0^R dU = \int_0^R 3g\frac{M^2(A,Z)}{R^6}r^4 dr = -\frac{3gM^2(A,Z)}{R^6}\frac{1}{5}r^5\Big|_0^R = -\frac{3gM^2(A,Z)}{5R}$$

$$\boxed{U = -3g\frac{M^2(A,Z)}{5R}}$$

2 p

(b) For a neutron star $Z = 0$, and $A \equiv N$. The gravitational field will give a positive contribution to the binding energy. Therefore, the expression of the binding energy will be the following:

$$B(N,0) = (a_V - a_A)N - a_S N^{2/3} + g\frac{3(NM_n)^2}{5R_S} =$$

$$= (a_V - a_A)N - a_S N^{2/3} + \frac{3}{5}g\frac{N^{5/3}M_n}{r_0}$$

where $R_S = r_0 N^{1/3}$

1 p

Two ways can be used for numerical solution, namely:

(A) complete calculation;

(B) introduction of the hypothesis $N \gg N^{2/3}$,for the neutron star.

0,5 p

In this second case the surface term vanish and the expression of the binding energy can be written as follows:

$$B(N,0) = (a_V - a_A)N + \frac{3}{5}g\frac{N^{5/3}M_n^2}{r_0}$$

Remark. Because $a_V < a_A$ there are situations where $B(N,0) < 0$, including in the hypothesis that N is very high and $N \gg N^{2/3}$; this is the most conveyable variant for an estimation.

The critical value of the neutron star radius can be obtained when the following condition is satisfied: $B(N,0) = 0$, namely:

$$B(N,0) = (a_V - a_A)N_{cr} + \frac{3}{5}g\frac{N_{cr}^{5/3}M_n^2}{r_0} = 0,$$

where N_{cr} is the neutron number for that the critical radius is obtained.

1,5 p

Solving the previous equation, the following results are obtained:

$$(a_A - a_V)N_{cr} = \frac{3}{5}g\frac{N_{cr}^{5/3}}{r_0}M_n^2 \ ,$$

$$\boxed{N_{cr}^{2/3} = \frac{(a_A - a_V)r_0}{3gM_n^2}}$$

The critical radius is $R_{cr} = r_0 N_{cr}^{1/3}$.

Calculations give the following values: $R_{cr} \cong 4345 \ m$, cu $N_{cr} \cong 4.81 x 10^{55}$.

VIII. Elementary particles

a) From Heisenberg relation $\Delta E \times \Delta \tau \sim \hbar$
If $\Delta E = \Gamma$, thus

$$\tau = \frac{\hbar}{\Gamma} = \frac{\hbar c}{\Gamma c} = \frac{197 \, MeV \, fm}{120 MeV \times 3 \times 10^8 \, m/s} = 5.5 \times 10^{-24} s$$

2,5 p

Lorentz factor is $\gamma = \frac{E}{m} = \frac{E_\Delta}{m_\Delta} = \frac{(1.232+200)}{1.232} \cong 163$

The distance traveled is: $d = \beta \gamma c \tau \cong 2.7 \times 10^{-13} m$

b) From the conservation law of the electric charge, $Q_X = +|e|$.
Baryon number $(B_\Delta = +1) = B_X + (B_\pi = 0) \rightarrow B_X = +1$
It is a strong decay; for Δ and pion the lepton numbers are all zero. Thus from the conservation of the lepton numbers result that for X particle the lepton number is zero. Similar arguments for strangeness.
For isospin:
$$\left(\vec{I}_\Delta = \frac{3}{2}; (I_\Delta)_3 = +\frac{3}{2}\right) \rightarrow \left(\vec{I}_X; (I_X)_3\right) + \left(\vec{I}_\pi = 1; (I_\pi)_3 = +1\right)$$
Using the rules for adding vector quantum numbers and their projections, $\rightarrow \left(\vec{I}_X = \frac{1}{2}; (I_X)_3 = +\frac{1}{2}\right)$

2,5 p

The X resonance is fermion.
At threshold, for the X particle (baryon) their mass must be bellow the value 1232-140=1092 MeV/c^2. The single positive baryon, with zero strange number and with mass <1090 MeV/c^2 is the proton.

c) Inverse Lorentz transformations give:

$$E_\pi^* = \gamma E_\pi (1 - \beta cos\theta) = E_\pi\left[\gamma - (\gamma^2 - 1)^{1/2} cos\theta\right]$$
$$\cong E_\pi\left[\gamma - \gamma\left(1 - \frac{1}{\gamma^2}\right)\left(1 - \frac{\theta^2}{2}\right)\right] \cong \left(\frac{E_\pi}{2\gamma}\right)(1 + \gamma^2\theta^2)$$

3 p

(where we have used the power development approximation and neglected terms in $\frac{\theta^2}{2\gamma}$ for $\gamma \gg 1$.)
But $E_\pi^* = (m_\Delta^2 - m_p^2)/2m_\Delta$ and $E_\Delta = \gamma m_\Delta$.

d) As quark structure: $\Delta^{++} \rightarrow (uuu)$, $p \rightarrow (uud)$, $\pi^+ \rightarrow (u\bar{d})$

1 p

IX. Rapdity and reference systems

I.6. The particle rapidity in the reference system S is: $y = \frac{1}{2}\ln\frac{E+p_L}{E-p_L}$; let be β the relative velocity of the particle in reference system LS. The relationships for energy and longitudinal momentum, $E = \gamma m$, $p_L = \gamma\beta m$, respectively, permit the writing of the new relationship between rapidities, namely:

$$y = \frac{1}{2}\ln\frac{\gamma m + \gamma\beta m}{\gamma m - \gamma\beta m} = \frac{1}{2}\ln\frac{1+\beta}{1-\beta} \quad (1) \qquad \textbf{3 p}$$

If the velocity of the particle, β, is small, then it is possible the following approximation: $y \cong \beta$.

When the particle moves with the relative velocity β on the direction z, parallel with the similar axis of the S' system moving in rapport with the S, the following expression of the rapidity can be written:

$$y' = \frac{1}{2}\ln\frac{E'+p'_L}{E'-p'_L}.(2) \qquad \textbf{1 p}$$

The Lorentz transformations between the two reference systems are the following:

$$E' = \gamma(E - \beta p_z)$$
$$p'_z = \gamma(p_z - \beta E) \qquad (3) \qquad\qquad \textbf{2 p}$$

Introducing the expressions from equations (3) in equation (2), the next relationship is obtained:

$$y' = \frac{1}{2}\ln\frac{\gamma(E-\beta p_L)+\gamma(p_L-\beta E)}{\gamma(E-\beta p_L)-\gamma(p_L-\beta E)} = \frac{1}{2}\ln\frac{E-\beta p_L+p_L-\beta E}{E-\beta p_L-p_L+\beta E} =$$
$$= \frac{1}{2}\ln\frac{E(1-\beta)+p_L(1-\beta)}{E(1+\beta)-p_L(1+\beta)} = \frac{1}{2}\ln\frac{(1-\beta)(E+p_L)}{(1+\beta)(E-p_L)} = \frac{1}{2}\ln\frac{E+p_L}{E-p_L}+\frac{1}{2}\ln\frac{1-\beta}{1+\beta} =$$
$$= y - \frac{1}{2}\ln\frac{1+\beta}{1-\beta} \qquad\qquad (4)$$

$$\textbf{2,5 p}$$

From the final relationship, $y' = y - \frac{1}{2}\ln\frac{1+\beta}{1-\beta}$, the final asked relationship is obtained, namely:

$$y' = y - y_\beta, \text{ with } y_\beta = \frac{1}{2}\ln\frac{1+\beta}{1-\beta} \qquad\qquad \textbf{0,5 p}$$

X.   Langmuir

1a) $\mu(T,P) = -k_B T \ln\left[\frac{k_B T}{P}\left(\frac{2\pi M}{h^2}k_B T\right)^{3/2}\right]$, **2 p**

1b) $\langle N_a\rangle(T,\mu;N_0) = \dfrac{N_a}{1+\left(\frac{h\omega}{2\pi k_B T}\right)^3 e^{-\varepsilon/k_B T}\ e^{-\mu/k_B T}}$ **2 p**

1c) $\theta(T,P) = \dfrac{1}{\frac{P_0(T)}{P}+1}$, $P_0(T) \equiv \left(\dfrac{M\omega^2}{2\pi}\right)^{3/2}\dfrac{e^{-\varepsilon/k_B T}}{\sqrt{k_B T}}$; **1 p**

2a) $\mu_g(T,V,N_g) = -k_B T \ln\left[\frac{V}{N_g}\left(\frac{2\pi M}{h^2}k_B T\right)^{3/2}\right]$

$= -k_B T \ln\left[\frac{k_B T}{P}\left(\frac{2\pi M}{h^2}k_B T\right)^{3/2}\right]$, **0,5 p**

2b) $\mu_a(T,N_a;N_0) = -k_B T \ln\left[\frac{N_0-N_a}{N_a}\left(\frac{2\pi}{h\omega}k_B T\right)^3 e^{\varepsilon/(k_B T)}\right]$, **2 p**

2c) $\dfrac{1}{\theta} = 1 + \dfrac{\frac{k_B T}{P}\left(\frac{2\pi M}{h^2}k_B T\right)^{3/2}}{\left(\frac{2\pi}{h\omega}k_B T\right)^3 e^{\varepsilon/(k_B T)}}$ $\Rightarrow$ $\theta = \dfrac{1}{\frac{P_0(T)}{P}+1}$, **1 p**

2d) The previous expressions of the covering ratio are identical, because the statistical ensembles are equivalent at thermodynamic limit. **0,5 p**