# 1 Classical Mechanics

## Spring me baby one more time!

Consider the system shown in Fig. 1, which is composed of two identical point masses $m$ connected by three identical massless springs of spring constant $k$. One spring connects the first mass to a fixed wall; another spring connects the two masses; and the third spring connects the second mass to another fixed wall. The motion is constrained to be one-dimensional along the horizontal axis.

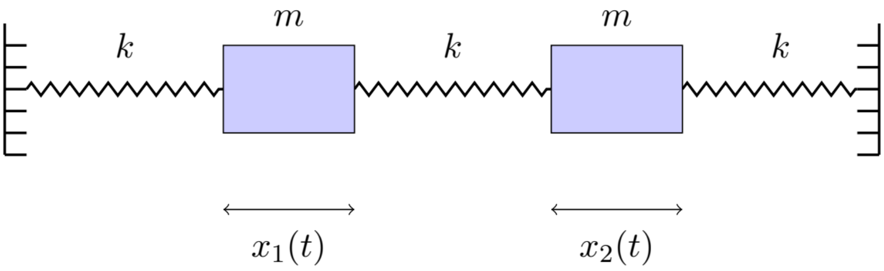

Figure 1: Two identical masses connected by three springs.

Let $x_1(t)$ and $x_2(t)$ denote the displacements of the two masses from equilibrium.

## QUESTIONS

1. **(1 point)** Write down the Lagrangian of the system.
2. **(1.5 points)** Derive the equations of motion for $x_1(t)$ and $x_2(t)$ by any method of your choice (e.g., the Newton's laws or the Lagrangian formalism).
3. **(1.5 points)** Find the normal modes of the system and their corresponding angular frequencies. Describe the physical motion associated with each mode.
4. **(1 point)** Suggest one set of initial conditions leading to regular (periodic) motion involving a single normal mode, and compute the corresponding total mechanical energy.
5. **(2 points)** Now suppose that the rightmost spring is slightly stiffer, with spring constant that is slightly distorted to $k(1+\varepsilon)$ where $|\varepsilon| \ll 1$, while the other two springs remain equal to $k$. Compute the corrections to the normal-mode angular frequencies up to first order in $\varepsilon$, and discuss briefly how the physical nature of the modes is affected.
6. **(2 points)** Assume now that the system with three identical springs ($\varepsilon = 0$) is in contact with a heat bath at temperature $T$, so that it is described by a canonical ensemble. Using the normal coordinates, compute the thermal averages $\langle x_1^2 \rangle$, $\langle x_2^2 \rangle$, and the mean total mechanical energy $\langle E \rangle$ of the system.
7. **(1 point)** How would the results of part (6) change qualitatively if the right spring constant were $k(1+\varepsilon)$ with $|\varepsilon| \ll 1$? (No explicit calculation is required; a qualitative answer is sufficient.)

## Solutions

1. **(1 point)** Write down the Lagrangian of the system.

   On the one hand, the kinetic energy of the system is

   $$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2. \tag{1}$$

   On the other hand, the potential energy stored in the springs is given by

   $$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}kx_2^2. \tag{2}$$

   Thus, the Lagrangian $\mathcal{L}$ is given by

   $$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k\left[x_1^2 + (x_2 - x_1)^2 + x_2^2\right]. \tag{3}$$

2. **(1.5 points)** Derive the equations of motion for $x_1(t)$ and $x_2(t)$ by any method of your choice (e.g., the Newton's laws or the Lagrangian formalism).

   The Euler-Lagrange equations associated with Eq. (3) are given by

   $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0, \tag{4}$$

   with $i = 1, 2$, which render the following equations of motion:

   $$m\ddot{x}_1 + 2kx_1 - kx_2 = 0, \tag{5}$$

   $$m\ddot{x}_2 + 2kx_2 - kx_1 = 0. \tag{6}$$

   Alternatively, one can also use Newton's second law. The forces acting on each of the masses equal

   $$F_1 = -kx_1 + k(x_2 - x_1), \tag{7}$$

   $$F_2 = -kx_2 - k(x_2 - x_1), \tag{8}$$

   and then, using $F = ma$, one finally obtains Eqs. (5) and (6).

3. **(1.5 points)** Find the normal modes of the system and their corresponding angular frequencies. Describe the physical motion associated with each mode.

   We look for solutions of the form

   $$x_j(t) = A_j e^{i\omega t}. \tag{9}$$

   Substituting into the equations of motion yields the system

   $$-m\omega^2 A_1 e^{i\omega t} + 2kA_1 e^{i\omega t} - kA_2 e^{i\omega t} = 0, \tag{10}$$

   $$-m\omega^2 A_2 e^{i\omega t} + 2kA_2 e^{i\omega t} - kA_1 e^{i\omega t} = 0, \tag{11}$$

   which equals

   $$-m\omega^2 A_1 + 2kA_1 - kA_2 = 0, \tag{12}$$

   $$-m\omega^2 A_2 + 2kA_2 - kA_1 = 0, \tag{13}$$

   Equations (12) and (13), which can be rewritten as

   $$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0. \tag{14}$$

Non-trivial solutions exist when the determinant vanishes

$$(2k - m\omega^2)^2 - k^2 = 0. \tag{15}$$

Solving for $\omega$, we find the two normal mode frequencies

$$\omega_1 = \sqrt{\frac{k}{m}}, \qquad \omega_2 = \sqrt{\frac{3k}{m}}, \tag{16}$$

whose normalized eigenvectors are given by

$$\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{u_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{17}$$

Therefore:

- For $\omega_1$, the eigenvector satisfies $A_1 = A_2$, and both masses oscillate *in phase.*
- For $\omega_2$, the eigenvector satisfies $A_1 = -A_2$, and the masses oscillate *out of phase.*

4. **(1 point)** Suggest one set of initial conditions leading to regular (periodic) motion involving a single normal mode, and compute the corresponding total mechanical energy.

A purely periodic motion is obtained by exciting a single normal mode. For example, choosing initial conditions

$$x_1(0) = A, \qquad x_2(0) = A, \qquad \dot{x}_1(0) = \dot{x}_2(0) = 0, \tag{18}$$

excites only the in-phase normal mode. The total mechanical energy for this motion is

$$E = V = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 = kA^2. \tag{19}$$

5. **(2 points)** Now suppose that the rightmost spring is slightly stiffer, with spring constant that is slightly distorted to $k(1+\varepsilon)$ where $|\varepsilon| \ll 1$, while the other two springs remain equal to $k$. Compute the corrections to the normal-mode angular frequencies up to first order in $\varepsilon$, and discuss briefly how the physical nature of the modes is affected.

Now the rightmost spring has constant $k(1+\varepsilon)$ with $|\varepsilon| \ll 1$, while the left and middle springs remain at $k$. The kinetic energy remains unchanged, and is given by Eq. (1). Conversely, the potential turns into

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(1+\varepsilon)x_2^2 = \frac{1}{2}(2k)\,x_1^2 - kx_1x_2 + \frac{1}{2}k(2+\varepsilon)\,x_2^2, \tag{20}$$

which can be written in matrix form as

$$V = \frac{1}{2}\begin{pmatrix} x_1 & x_2 \end{pmatrix} K \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad K = \begin{pmatrix} 2k & -k \\ -k & (2+\varepsilon)k \end{pmatrix}, \tag{21}$$

with the mass matrix given by

$$M = m\mathbb{I}, \tag{22}$$

and

$$K = K^{(0)} + \Delta K, \qquad K^{(0)} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}, \qquad \Delta K = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon k \end{pmatrix}. \tag{23}$$

Then, the normal-mode frequencies satisfy

$$\det\left(K - \lambda\mathbb{I}\right) = 0, \tag{24}$$

with $\lambda = m\omega^2$.

For $\varepsilon = 0$, Eq. (24) reduces to Eq. (15). As a consequence,

$$\lambda_1^{(0)} = k, \qquad \lambda_2^{(0)} = 3k, \tag{25}$$

with the normalized eigenvectors given by Eq. (17), and the normal mode frequencies provided by Eq. (16)

$$\omega_1^{(0)} = \sqrt{\frac{k}{m}}, \qquad \omega_2^{(0)} = \sqrt{\frac{3k}{m}}. \tag{26}$$

The first-order corrections to the eigenvalues can be computed using non-degenerate perturbation theory, rendering

$$\delta\lambda_j = \left(\mathbf{u_j}^{(0)}\right)^T \Delta K \, \mathbf{u_j}. \tag{27}$$

The first order correction to the first normal mode of frequency $\omega_1^{(0)}$ depends on the perturbed term in the eigenvalue

$$\delta\lambda_1 = \mathbf{u_1}^T \Delta K \, \mathbf{u_1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon k \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \varepsilon k \end{pmatrix} = \frac{1}{2} \varepsilon k. \tag{28}$$

Then

$$\lambda_1 \approx \lambda_1^{(0)} + \delta\lambda_1 = k + \frac{\varepsilon k}{2}, \tag{29}$$

and the new perturbed frequency equals

$$\omega_1^2 = \frac{\lambda_1}{m} \approx \frac{1}{m}\left(k + \frac{\varepsilon k}{2}\right) = \frac{k}{m}\left(1 + \frac{\varepsilon}{2}\right), \tag{30}$$

which can be approximated up to first order in $\varepsilon$ by

$$\omega_1 \approx \sqrt{\frac{k}{m}}\left(1 + \frac{\varepsilon}{4}\right). \tag{31}$$

Similarly, the frequency $\omega_2^{(0)}$ of the second normal mode of depends on the corresponding perturbation in the eigenvalue

$$\delta\lambda_2 = \mathbf{u_2}^T \Delta K \, \mathbf{u_2} = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon k \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -\varepsilon k \end{pmatrix} = \frac{1}{2} \varepsilon k. \tag{32}$$

Thus,

$$\lambda_2 \approx 3k + \frac{\varepsilon k}{2}, \qquad \omega_2^2 \approx \frac{1}{m}\left(3k + \frac{\varepsilon k}{2}\right) = \frac{k}{m}\left(3 + \frac{\varepsilon}{2}\right), \tag{33}$$

and to first order in $\varepsilon$,

$$\omega_2 \approx \sqrt{\frac{3k}{m}}\left(1 + \frac{\varepsilon}{12}\right). \tag{34}$$

**Physical interpretation.** The eigenvectors are only slightly modified (at order $\varepsilon$), so, to first order, the modes remain approximately in-phase and out-of-phase oscillations. Both frequencies increase because stiffening the right spring raises the overall restoring forces. The relative increase is modest and different for each mode: the mode involving comparatively larger motion of the second mass is affected more strongly in higher orders, but at first order both eigenvalues receive the same additive correction $\delta\lambda_n = \varepsilon k/2$.

6. **(2 points)** Assume now that the system with three identical springs ($\varepsilon = 0$) is in contact with a heat bath at temperature $T$, so that it is described by a canonical ensemble. Using the normal coordinates, compute the thermal averages $\langle x_1^2 \rangle$, $\langle x_2^2 \rangle$, and the mean total mechanical energy $\langle E \rangle$ of the system.

We now return to the symmetric case with three identical springs ($\varepsilon = 0$). The system has two quadratic degrees of freedom and can be diagonalized in terms of normal coordinates. Define the normal coordinates $Q_1, Q_2$ (and their velocities) as

$$Q_1 = \frac{x_1 + x_2}{\sqrt{2}}, \qquad Q_2 = \frac{x_1 - x_2}{\sqrt{2}}. \tag{35}$$

This transformation is orthogonal, so the kinetic energy becomes

$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2 = \frac{1}{2} m\dot{Q}_1^2 + \frac{1}{2} m\dot{Q}_2^2. \tag{36}$$

From the analysis of the normal modes, we know that the potential energy can be written as

$$V = \frac{1}{2} m\omega_1^2 Q_1^2 + \frac{1}{2} m\omega_2^2 Q_2^2, \tag{37}$$

with $\omega_1$ and $\omega_2$ given by Eq. (16). Hence the total energy is

$$E = \frac{1}{2} m\dot{Q}_1^2 + \frac{1}{2} m\omega_1^2 Q_1^2 + \frac{1}{2} m\dot{Q}_2^2 + \frac{1}{2} m\omega_2^2 Q_2^2. \tag{38}$$

In the canonical ensemble at temperature $T$, each quadratic term in the Hamiltonian contributes $\frac{1}{2} k_B T$ on average (equipartition theorem). Thus, for each normal mode,

$$\left\langle \frac{1}{2} m\dot{Q}_i^2 \right\rangle = \frac{1}{2} k_B T, \qquad \left\langle \frac{1}{2} m\omega_i^2 Q_i^2 \right\rangle = \frac{1}{2} k_B T, \tag{39}$$

with $i = 1, 2$. In particular,

$$\langle Q_i^2 \rangle = \frac{k_B T}{m\omega_i^2}, \tag{40}$$

with $i = 1, 2$. Using the explicit values of the frequencies,

$$\langle Q_1^2 \rangle = \frac{k_B T}{m(k/m)} = \frac{k_B T}{k}, \qquad \langle Q_2^2 \rangle = \frac{k_B T}{m(3k/m)} = \frac{k_B T}{3k}. \tag{41}$$

Now, express $x_1$ and $x_2$ in terms of $Q_1$ and $Q_2$. The inverse transformation is

$$x_1 = \frac{Q_1 + Q_2}{\sqrt{2}}, \qquad x_2 = \frac{Q_1 - Q_2}{\sqrt{2}}. \tag{42}$$

We assume that in equilibrium the cross-correlation $\langle Q_1 Q_2 \rangle = 0$, since the normal modes are statistically independent in the canonical ensemble. Then

$$\langle x_1^2 \rangle = \frac{1}{2} \left( \langle Q_1^2 \rangle + 2\langle Q_1 Q_2 \rangle + \langle Q_2^2 \rangle \right) = \frac{1}{2} \left( \langle Q_1^2 \rangle + \langle Q_2^2 \rangle \right), \tag{43}$$

$$\langle x_2^2 \rangle = \frac{1}{2} \left( \langle Q_1^2 \rangle - 2\langle Q_1 Q_2 \rangle + \langle Q_2^2 \rangle \right) = \frac{1}{2} \left( \langle Q_1^2 \rangle + \langle Q_2^2 \rangle \right). \tag{44}$$

Therefore,

$$\langle x_1^2 \rangle = \langle x_2^2 \rangle = \frac{1}{2} \left( \frac{k_B T}{k} + \frac{k_B T}{3k} \right) = \frac{k_B T}{2k} \left( 1 + \frac{1}{3} \right) = \frac{2}{3} \frac{k_B T}{k}. \tag{45}$$

Finally, the mean total mechanical energy is

$$\langle E\rangle = \sum_{i=1}^{2}\left(\left\langle \frac{1}{2}m\dot{Q}_i^2\right\rangle + \left\langle \frac{1}{2}m\omega_i^2 Q_i^2\right\rangle\right) = 2\times\left(\frac{1}{2}k_BT+\frac{1}{2}k_BT\right) = 2k_BT, \tag{46}$$

a results that is consistent with the equipartition theorem as for a system with two degrees of freedom and two quadratic terms (kinetic and potential) per degree of freedom, $\langle E\rangle = (\text{number of quadratic terms})\times\frac{1}{2}k_BT = 4\times\frac{1}{2}k_BT = 2k_BT$.

7. **(1 point)** How would the results of part (6) change qualitatively if the right spring constant were $k(1+\varepsilon)$ with $|\varepsilon| \ll 1$? (No explicit calculation is required; a qualitative answer is sufficient.)

A small $\varepsilon$ modifies the two frequencies by the shifts computed in part (4). Since equipartition gives

$$\langle Q_i^2\rangle = \frac{k_BT}{m\omega_i^2}, \tag{47}$$

a slight increase in either $\omega_i$ decreases the corresponding $\langle Q_i^2\rangle$. As a consequence:

- The more strongly the normal mode involves motion of mass 2, the more suppressed its thermal amplitude becomes.
- The mean energy remains $\langle E\rangle = 2k_BT$ (equipartition depends on the number of quadratic terms, not their coefficients).

## 2 Gravitation

# Relativistic Time Corrections

An artificial satellite orbits the Earth on the equatorial plane with a perigee of 7,125 km and an apogee of 9,232 km (orbit called $Orb_0$; see Fig. 1, left). The satellite is equipped with an atomic clock that generates signals sent to the Earth. On Earth, there are two research stations that keep track of these signals, one at the South Pole and another one at the Equator.

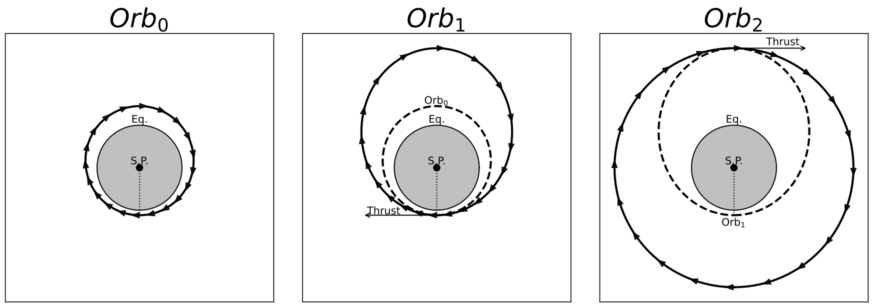

Figure 1: **Left:** initial orbit ($Orb_0$, moving clockwise), with the Earth shown in grey (the South Pole and the Equator are marked as **S.P.** and **Eq.**, respectively). The dotted line marks the direction to the perigee. **Center:** orbit $Orb_1$, achieved when accelerating (with a thrust) at the perigee of $Orb_0$. The old orbit $Orb_0$ is shown as a dashed line, for reference. **Right:** orbit $Orb_2$, achieved after accelerating at the apogee of $Orb_1$ (the orbit $Orb_1$ is shown as a dashed line, also for reference).

## QUESTIONS

1. **(1 point)** Assuming Newton's Mechanics, compute all the parameters of $Orb_0$ (Fig. 1, left).
2. **(1 point)** When the satellite crosses the perigee of $Orb_0$, its speed is increased by 1,000 m/s, bringing it into a new orbit, $Orb_1$ (Fig. 1, center). Afterwards, the satellite is put into a circular orbit, $Orb_2$, with a radius equal to that of the apogee of $Orb_1$ (Fig. 1, right). What is the period of $Orb_2$?
3. Given one second of coordinate time, $\Delta t = 1$ s, compute:
   (a) **(2 points)** The difference between $\Delta t$ and the proper times registered at the South Pole, $\Delta\tau_{SP}$, and at the Equator, $\Delta\tau_{Eq}$ [use Eq. (2)]. Compute also $\Delta\tau_{SP} - \Delta\tau_{Eq}$.
   (b) **(2 points)** The difference between $\Delta t$ and the time elapsed at the satellite in $Orb_2$, $\Delta\tau_S$.
   (c) **(1 point)** The velocity of the observer at the Equator, $v_{Eq}$, referred to the South Pole. Use this velocity to compute the difference (after $\Delta t = 1$ s) between time lapses at the South Pole and Equator, $\Delta\tau_{SP} - \Delta\tau_{Eq}$, according to the Theory of Special Relativity. Compare it to the result of Question 3(a), based on General Relativity, and briefly comment on this comparison.
4. **(3 points)** The orbital period $T_0$ computed in Question 1 is given in the frame of a fixed distant observer (i.e., it is a *coordinate time*, $t$). Knowing this, compare it with the orbital period $T_0^{SP}$, measured at the South Pole, and with the period $T_0^S$, measured by the atomic clock on board of the satellite, assuming a Keplerian orbit. Given that $T_0^{SP} - T_0^S \neq 0$, the satellite is continuously loosing synchronization with the clock at the South Pole. How much time is needed for the synchronization to be off by 1 ns?

### Information & Hints

Under the assumptions of the Theory of Special Relativity, the speed of the observer at the Equator referred to the South Pole, $v_{Eq}$, causes a time dilation of the former when compared to the time for the later. If $\Delta\tau_{Eq}$ and $\Delta t_{SP}$ are time lapses at the Equator and the South Pole, respectively, Special Relativity tells us that:

$$\Delta\tau_{SP} = \frac{\Delta\tau_{Eq}}{\sqrt{1 - \frac{v_{Eq}^2}{c^2}}}. \tag{1}$$

Regarding the satellite, the advance of proper time, $d\tau$, registered by its atomic clock is affected by General Relativity and can be approximated as:

$$c^2 d\tau^2 \approx \left(1 - \frac{2GM_E}{c^2 r}\right) c^2 dt^2 - \left(1 + \frac{2GM_E}{c^2 r}\right) dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2, \tag{2}$$

where $t$ is the time measured by an observer (at rest) very far from Earth (this is the so-called *coordinate time*) and $r$, $\theta$, and $\phi$ are the usual spherical coordinates. In Eq. (2), we have assumed $r \gg r_s = 2GM_E/c^2$, where $r_s$ is the Schwarzschild radius for an Earth mass, $M_E$. This equation can also be used to estimate the time dilation, $d\tau/dt$, for any observer standing on the Earth's surface, by doing:

$$r = R_E, \quad \theta = \theta_0, \quad dr = d\theta = 0, \quad \frac{d\phi}{dt} = 2\pi \text{ day}^{-1}, \tag{3}$$

where $\theta_0$ is the observer's latitude and $R_E$ the Earth radius.

### Additional considerations:

- We assume that Earth is a perfect homogeneous sphere ($M_E = 5.97 \times 10^{24}$ kg, $R_E = 6{,}370$ km). The Newton constant is $G = 6.67 \times 10^{-11}$ m$^3$·kg$^{-1}$·s$^{-2}$.
- Third Kepler's Law for the satellite: $T^2 = \dfrac{4\pi^2 a^3}{GM_E}$, with $a = \dfrac{r_a + r_p}{2}$.
- When adding velocities, you can assume Galilean Relativity.
- For a closed Keplerian orbit: $r = \dfrac{a(1-e^2)}{1 + e\cos\theta}$, with $e = \dfrac{r_{max} - r_{min}}{r_{max} + r_{min}}$.
- The radial velocity in a closed Keplerian orbit on the Earth's equatorial plane is: $\dot{r} = \sqrt{\dfrac{GM_E}{a(1-e^2)}}\, e\sin\theta$, where $\theta$ is defined with its origin at the perigee.
- $(1 + kx)^{-1} \approx 1 - kx + k^2x^2 - k^3x^3 + \cdots$
- If $v \ll c$ (a well satisfied condition), then $dt^2 - d\tau^2 = (dt + d\tau)(dt - d\tau) \approx 2dt(dt - d\tau)$.
- $T_0 - T_0^S = \displaystyle\int_0^{T_0} \frac{dt - d\tau_S}{dt}\, dt$.

# Solutions

1. **(1 point)** Assuming Newton's Mechanics, compute all the parameters of $Orb_0$ (Fig. 1, left).

   The total mechanical energy, $E_0$, of the satellite at $Orb_0$, computed at the apogee and perigee (and using the gauge for the gravitational potential at infinity) is

   $$E_0 = \frac{1}{2}mv_{p0}^2 - \frac{GmM_E}{r_{p0}} = \frac{1}{2}mv_{a0}^2 - \frac{GmM_E}{r_{a0}}, \tag{4}$$

   where $r_{a0}$ and $v_{a0}$ are the orbital distance and geocentric velocity of $Orb_0$ at the apogee, and $r_{p0}$ and $v_{p0}$ are those at the perigee. On the other hand, when the orbital distance is minimum or maximum, the geocentric position vector is perpendicular to the velocity. Therefore, the modulus of the angular momentum, $L_0$, can be easily computed as

   $$L_0 = mr_{a0}v_{a0} = mr_{p0}v_{p0}. \tag{5}$$

   Combining both equations allows us to find a relation between distances at apogee and perigee and the respective satellite speeds:

   $$\frac{1}{2}\left(\frac{r_{p0}^2}{r_{a0}^2} - 1\right)v_{p0}^2 = GM_E\left(\frac{1}{r_{a0}} - \frac{1}{r_{p0}}\right). \tag{6}$$

   Just substituting on Eq. (6), we can obtain the required velocities:

   - $v_{a0} = 7{,}942.7$ m/s.
   - $v_{p0} = 6{,}130.0$ m/s.

   Now, we have all the information to compute $E_0$ from Eq. (4) and $L_0$ from Eq. (5). The results are:

   - $E_0 = -7.3033 \times 10^9$ J (potential gauge at infinity).
   - $L_{p0} = 16.978 \times 10^{12}$ kg·m$^2$·s$^{-2}$.

   Finally, using the Third Kepler's Law (for the case of the Earth as the central body), the period is straightforward to compute:

   - $T_0 = 2\pi\sqrt{\frac{1}{GM_E}\left(\frac{r_{a0}+r_{p0}}{2}\right)^3} = 7{,}364.4$ s.

   The semi-major axis and eccentricity are obtained easily:

   - $a_0 = \frac{r_{a0}+r_{p0}}{2} = 8.179 \times 10^6$ m.
   - $e_0 = \frac{r_{a0}-r_{p0}}{r_{a0}+r_{p0}} = 0.1288$.

2. **(1 point)** When the satellite crosses the perigee of $Orb_0$, its speed is increased by 1,000 m/s, bringing it into a new orbit, $Orb_1$ (Fig. 1, center). Afterwards, the satellite is put into a circular orbit, $Orb_2$, with a radius equal to that of the apogee of $Orb_1$ (Fig. 1, right). What is the period of $Orb_2$?

   In the new orbit, $Orb_1$, we know the distance and speed at one of its points (which is, indeed, the perigee of $Orb_1$, coinciding with the perigee or $Orb_0$):

   $$r_{p1} = r_{p0} = 7.125 \times 10^6 \text{ m}, \quad v_{p1} = v_{p0} + 1{,}000 \text{ m/s}. \tag{7}$$

We can solve for this new orbit by combining, again, Eqs. (4) and (5), so that we can deduce, for instance, $r_a$ (or $r_p$) as a function of $v_p$ and $r_p$ (or $r_a$). It is not difficult [from Eq. (6)] to get the relation:

$$\frac{r_{p1}^2 v_{p1}^2}{2}\left(\frac{1}{r_{a1}}\right)^2 - GM_E\frac{1}{r_{a_1}} - E_1 = 0, \tag{8}$$

where $E_1$ is the new mechanical energy (it can be computed easily, since we know the position and velocity at the perigee of the new orbit). This is a second-order equation for the inverse of $r_{a1}$, which we can solve and use it to retrieve all the orbital parameters[1], even though we are not asked to compute them:

- $r_{a1} = 17.917 \times 10^6$ m, $r_{p1} = r_{p0} = 7.125 \times 10^6$ m.
- $a_1 = \dfrac{r_a + r_p}{2} = 12.521 \times 10^6$ m.
- $T_1 = 13{,}950.1$ s (with 3rd Kepler's Law).
- $e_1 = \dfrac{r_a - r_p}{r_a + r_p} = 0.4309$
- $E_1 = -4.7705 \times 10^9$ J.
- $L_1 = 19.115 \times 10^{12}$ kg·m$^2$·s$^{-2}$.

A circular orbit, $Orb_2$, with radius $r_2 = r_{a1}$, would have a period (using Kepler's Third Law)

$$T_2 = 2\pi\sqrt{\frac{r_2^3}{GM_E}} = 23{,}878.7 \text{ s}. \tag{9}$$

3. Given one second of coordinate time, $\Delta t = 1$ s, compute:

   (a) **(2 points)** The difference between $\Delta t$ and the proper times registered at the South Pole, $\Delta\tau_{SP}$, and at the Equator, $\Delta\tau_{Eq}$ [use Eq. (2)]. Compute also $\Delta\tau_{SP} - \Delta\tau_{Eq}$.

   For the observer at the South Pole (i.e., at $\theta = 0$), Eq. (2) simplifies to

$$\Delta t^2 - \Delta\tau_{SP}^2 = \frac{2GM_E}{c^2 R_E}\,\Delta t^2. \tag{10}$$

   Now, using some algebra and the good approximation $\Delta t + \Delta\tau_{SP} \sim 2\Delta t$, we have:

$$\Delta t^2 - \Delta\tau_{SP}^2 = (\Delta t + \Delta\tau_{SP})(\Delta t - \Delta\tau_{SP}) \approx 2\Delta t(\Delta t - \Delta\tau_{SP}) = \frac{2GM_E}{c^2 R_E}\,\Delta t^2. \tag{11}$$

   That is (with $\Delta t = 1$ s):

$$\Delta t - \Delta\tau_{SP} = \frac{GM_E}{c^2 R_E}\,\Delta t = 6.95536 \times 10^{-10} \text{ s}. \tag{12}$$

   Another way to attack this problem is with the square root of the metric, i.e.:

$$\Delta\tau_{SP} = \sqrt{1 - \frac{2GM_E}{c^2 R_E}}\,\Delta t \;\rightarrow\; \Delta t - \Delta\tau_{SP} = \left(1 - \sqrt{1 - \frac{2GM_E}{c^2 R_E}}\right)\Delta t, \tag{13}$$

   which is very similar to Eq. (12) (actually, the former is equal to the first-order Taylor expansion of the square root in the later). Now, for the other observer at the Equator (and doing a very similar algebra as above):

$$\Delta t - \Delta\tau_{Eq} = \frac{GM_E}{c^2 R_E}\,\Delta t + \frac{R_E^2}{2c^2}\left(\frac{2\pi}{86{,}400}\right)^2 = 6.96729 \times 10^{-10} \text{ s}. \tag{14}$$

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[1] This equation has two solutions; one of them provides $r_{a1}$, while the other gives us, indeed, $r_{p1}$, which is the value that we used as the input!

Notice that $\Delta\phi$ is numerically equal to $\dot{\phi}$ in SI units, since $\Delta t = 1$ s. The difference between times at the South Pole and the Equator [based on Eq. (2)] is, thus:

$$\Delta\tau_{SP} - \Delta\tau_{Eq} = 1.194 \times 10^{-12} \text{ s.} \tag{15}$$

(b) **(2 points)** The difference between $\Delta t$ and the time elapsed at the satellite in $Orb_2$, $\Delta\tau_S$.

When the satellite is in $Orb_2$, Eq. (2) takes the form:

$$\Delta t - \Delta\tau_S = \frac{GM_E}{c^2 r_2}\Delta t + \frac{r_2^2}{2c^2}\left(2\pi\frac{\Delta t}{T_2}\right)^2, \tag{16}$$

where $d\phi$ is easily derived from the fact that $Orb_2$ is circular (i.e., uniform angular velocity). Notice that the circularity also implies $\Delta r = 0$ [thus simplifying Eq. (2) notably]. The final result for the time elapsed at the satellite is:

- $\Delta t - \Delta\tau_S = 4.9458 \times 10^{-10}$ s.

(c) **(1 point)** The velocity of the observer at the Equator, $v_{Eq}$, referred to the South Pole. Use this velocity to compute the difference (after $\Delta t = 1$ s) between time lapses at the South Pole and Equator, $\Delta\tau_{SP} - \Delta\tau_{Eq}$, according to the Theory of Special Relativity. Compare it to the result of Question 3(a), based on General Relativity, and briefly comment on this comparison.

The velocity of any point at the Equator, measured with respect to the South Pole, is

$$v_{Eq} = \frac{2\pi R_E[m]}{86400} \text{ m/s} = 463.2 \text{ m/s.} \tag{17}$$

Expanding the Lorentz equation, Eq. (1), to a second-order Taylor series (a good approximation to make things easier), we have:

$$\Delta\tau_{SP} - \Delta\tau_{Eq} \sim \frac{v_{Eq}^2}{2c^2}\Delta\tau_{Eq} = 1.194 \times 10^{-12} \text{ s.} \tag{18}$$

This is *the same* result that we found in part (a) of this exercise (isn't it beautiful?). This means that, to a very good approximation, Special Relativity (i.e., just accounting for the relative speeds between observers at the South Pole and the Equator) gives the same result as General Relativity (i.e., accounting for the whole curved spacetime metric). How is this possible? The answer is, actually, quite simple. The metric in Eq. (2) has spherical symmetry, and we are also assuming such a symmetry for the Earth surface. Therefore, all the frames defined on the Earth surface are experiencing the same spacetime curvature, so that their *relative times* can be computed using simple Special Relativity. However, this simplification does not hold when we compare times among observers at different distances to the Earth center (as it is the case of the satellite).

4. **(3 points)** The orbital period $T_0$ computed in Question 1 is given in the frame of a fixed distant observer (i.e., it is a *coordinate time*, $t$). Knowing this, compare it with the orbital period $T_0^{SP}$, measured at the South Pole, and with the period $T_0^S$, measured by the atomic clock on board of the satellite, assuming a Keplerian orbit. Given that $T_0^{SP} - T_0^S \neq 0$, the satellite is continuously loosing synchronization with the clock at the South Pole. How much time is needed for the synchronization to be off by 1 ns?

The first part of this exercise is straightforward. From exercise 3, we know how to compute the difference of time lapses between a distant observer and the South Pole. Therefore, we deduce that

$$T_0 - T_0^{SP} = \frac{GM_E}{c^2 R_E}T_0 = 5.1222 \times 10^{-6} \text{ s.} \tag{19}$$

That's it! However, the last part of the exercise (i.e., computing the time lapse on board of the satellite) is much more difficult. Manipulating Eq. (2), in a similar way as we did in exercise 3, we arrive to:

$$dt - d\tau_S \sim \frac{GM_E}{c^2 r}\,dt + \frac{1}{2c^2}\left(1 + \frac{2GM_E}{c^2 r}\right)\frac{dr}{dt}\,dr + \frac{r^2}{2c^2}\frac{d\phi}{dt}\,d\phi. \tag{20}$$

Dividing by $dt$, we get:

$$\frac{dt - d\tau_S}{dt} \sim \frac{GM_E}{c^2 r} + \frac{1}{2c^2}\left(1 + \frac{2GM_E}{c^2 r}\right)\left(\frac{dr}{dt}\right)^2 + \frac{r^2}{2c^2}\left(\frac{d\phi}{dt}\right)^2. \tag{21}$$

And now, integrating in time over a whole orbital period, we obtain, for the left-hand side of the equation:

$$\int_0^{T_0} \frac{dt - d\tau_S}{dt}\,dt = T_0 - T_0^S, \tag{22}$$

which is the quantity that we want to compute. Regarding the right-hand side of the equation, we have:

$$T_0 - T_0^S \sim \int_0^{T_0} \frac{GM_E}{c^2 r}\,dt + \frac{1}{2c^2}\int_0^{T_0}\left(1 + \frac{2GM_E}{c^2 r}\right)\left(\frac{dr}{dt}\right)^2 dt + \frac{1}{2c^2}\int_0^{T_0} r^2\left(\frac{d\phi}{dt}\right)^2 dt. \tag{23}$$

Let us work out these integrals one by one. We will give them the following names:

$$T_0 - T_0^S \sim I_t + I_r + I_\phi. \tag{24}$$

**Integral $I_t$:**
Changing the integration variable, from $t$ to $\phi$, and using the conservation of angular momentum:

$$I_t = \int_0^{T_0} \frac{GM_E}{c^2 r}\,dt = \int_0^{2\pi} \frac{GM_E}{c^2 r}\frac{dt}{d\phi}\,d\phi = \int_0^{2\pi} \frac{GM_E}{c^2 r}\frac{mr^2}{L}\,d\phi. \tag{25}$$

Now, using the equation for the elliptical orbit:

$$I_t = \frac{GmM_E a_0(1 - e_0^2)}{L_0 c^2}\int_0^{2\pi} \frac{1}{1 + e_0\cos\phi}\,d\phi. \tag{26}$$

Even though this integral has an exact solution (just look for it in a good table of definite integrals), we can approximate it with a Taylor expansion, using the eccentricity, $e_0 \sim 0$, as the expansion variable. Going to second-order Taylor:

$$I_t \approx \frac{GmM_E a_0(1 - e_0^2)}{L_0 c^2}\int_0^{2\pi}\left(1 - e_0\cos\phi + e_0^2\cos^2\phi\right)d\phi = \frac{\pi GmM_E a_0(1 - e_0^2)(2 + e_0^2)}{L_0 c^2}. \tag{27}$$

This expression gives us the result $I_t = 3.9891 \times 10^{-6}$ s, which is indeed quite similar to the one obtained by solving the exact integral (i.e., $I_t = 3.9896 \times 10^{-6}$ s).

**Integral $I_r$:**
Using the equation for the radial velocity in a Keplerian orbit, and changing variables from $t$ to $\phi$, we get:

$$I_r = \frac{1}{2c^2}\int_0^{2\pi}\left(1 + \frac{2GM_E}{c^2 r}\right)\frac{GM_E}{a(1 - e^2)}(e\sin\phi)^2\frac{dt}{d\phi}\,d\phi. \tag{28}$$

Elaborating a bit:

$$I_r = \frac{1}{2c^2}\frac{e_0^2 GmM_E}{a_0(1 - e_0^2)L_0}\int_0^{2\pi}\left(1 + \frac{2GM_E}{c^2 r}\right)(\sin\phi)^2 r^2 d\phi. \tag{29}$$

This integral can be divided in two parts:

$$I_r = \frac{1}{2c^2}\frac{e_0^2 GmM_E}{a_0(1-e_0^2)L_0}\left[\int_0^{2\pi}(\sin\phi)^2 r^2 d\phi + \int_0^{2\pi}\frac{2GM_E}{c^2}(\sin\phi)^2 r d\phi\right], \tag{30}$$

which, after writing $r$ as a function of $\phi$, become:

$$I_r = \frac{e_0^2 GmM_E}{2L_0c^2}\left[a_0(1-e_0^2)\int_0^{2\pi}\left(\frac{\sin\phi}{1+e_0\cos\phi}\right)^2 d\phi + \frac{2GM_E}{c^2}\int_0^{2\pi}\frac{(\sin\phi)^2}{1+e_0\cos\phi}d\phi\right], \tag{31}$$

We can also approximate the denominators of the integrands in a similar way as before (going, for instance, to a second-order Taylor expansion), so that we reduce the integrals to powers of trigonometric functions. In that case, the result is (after a somewhat tedious integration):

$$I_r \approx \frac{e_0^2 GmM_E}{2L_0c^2}\left[a_0(1-e_0^2)\frac{\pi(e_0^4+6e_0^2+8)}{8} + \frac{2GM_E}{c^2}\frac{\pi(e_0^2+4)}{4}\right] = 1.662\times10^{-8}\text{s}, \tag{32}$$

which is about two orders of magnitude smaller than $I_t$.

**Integral $I_\phi$:**
This integral is, by far, the easiest one. We proceed with the same strategy, by changing the integration variable from $t$ to $\phi$ with the chain rule:

$$I_\phi = \frac{1}{2c^2}\int_0^{2\pi} r^2\left(\frac{d\phi}{dt}\right)^2\frac{dt}{d\phi}d\phi = \frac{1}{2c^2}\int_0^{2\pi} r^2\frac{L_0}{mr^2}d\phi = \frac{\pi L_0}{mc^2}. \tag{33}$$

That's it! The result is: $I_\phi = 1.9782\times10^{-6}$ s, which is of the same order of magnitude of $I_t$.

**Total result:**
Adding up the three integrals, we can finally compute the difference between the orbital period measured by the distant observer and by the atomic clock on board of the satellite. We notice that the precession of the perigee (a pure relativistic effect) is not considered here, since we are assuming a Keplerian orbit for the satellite:

$$T_0 - T_0^S = 5.9839\times10^{-6}\text{ s}. \tag{34}$$

This is almost 6 $\mu$s! If we compare it now with the period measured at the South Pole, we get:

$$T_0^{SP} - T_0^S = 0.8617\times10^{-6}\text{ s}. \tag{35}$$

To accumulate a proper-time difference of $d\tau_{S-SP} = 1$ ns between the satellite and the South Pole, we need a time lapse, $\Delta t$, of

$$\Delta t = \frac{T_0}{T_0^{SP}-T_0^S}\times d\tau_{S-SP} = 8.55\text{ s}. \tag{36}$$

As we can see, if we want to keep a good synchronization between the South Pole (or, in fact, any other clock on the Earth surface) and the satellite, we have to apply time corrections relatively fast.

## 3 Atmospheric & Oceanic Physics

## The Planck Feedback

The Earth's surface temperature is not merely a result of a static greenhouse "blanket", but is determined by the altitude at which the atmosphere becomes transparent to long-wave radiation. This problem explores the concept of the *emission level* and how greenhouse gas (GHG) concentrations influence the planetary energy balance.

### QUESTIONS

1. Consider a simplified model where the Earth's surface emits as a black body at temperature $T_s$. The atmosphere is represented by a single isothermal layer at temperature $T_a$. This layer absorbs a fraction $\epsilon$ of the infrared radiation (emissivity) and allows the remaining $(1-\epsilon)$ to escape directly to space through the so-called "atmospheric window".
   (a) **(2 points)** Write the energy balance equations for (i) the atmosphere, and (ii) the Earth-atmosphere system as a whole, assuming radiative equilibrium. Use $S_0$ for the solar constant and $A$ for the planetary albedo.
   (b) **(2 points)** Prove that the surface temperature $T_s$ can be expressed as:
   $$T_s = T_e \left(\frac{2}{2-\epsilon}\right)^{1/4}, \tag{1}$$
   where $T_e$ is the effective emission temperature of the planet. Calculate $T_s$ given $T_e = 255$ K and $\epsilon = 0.77$.

2. **(2 points)** The increase in GHG concentrations effectively "closes" parts of the infrared spectrum. For $CO_2$, the radiative forcing ($\Delta F$, the change in net flux at the top of the atmosphere) follows a logarithmic law:
   $$\Delta F = \alpha \ln\left(\frac{C}{C_0}\right), \tag{2}$$
   where $\alpha = 5.35\ \mathrm{W/m^{-2}}$ and $C/C_0$ is the ratio of final to initial $CO_2$ concentration. Calculate the forcing $\Delta F$ for a doubling of $CO_2$ concentration ($C/C_0 = 2$). Using the Stefan-Boltzmann law calculate the required change in the effective emission temperature ($\Delta T_e$) to restore equilibrium after this forcing, assuming $T_e$ was initially 255 K. Use $\sigma = 5.67 \times 10^{-8}\ \mathrm{W{\cdot}m^{-2}{\cdot}K^{-4}}$.

3. **(2 points)** In the real atmosphere, heat transport is not solely radiative but also convective, maintaining a vertical lapse rate $\Gamma = -dT/dz \approx 6.5$ K/km. The radiation escaping to space is emitted, on average, from an effective altitude $z_e$ such that $T(z_e) = T_e$. As GHG concentrations increase, the atmosphere becomes more opaque, shifting the emission level $z_e$ upwards. Show that if $z_e$ increases by $\Delta z_e$, the surface temperature $T_s$ must increase by $\Delta T_s = \Gamma \Delta z_e$ to maintain the same $T_e$ (and thus restore radiative balance). If a doubling of $\mathrm{CO_2}$ raises the emission level $z_e$ by 150 m, calculate the resulting increase in surface temperature $\Delta T_s$.

4. In the previous Question, we assumed the lapse rate $\Gamma$ remains constant. However, in a warming atmosphere, increased evaporation leads to more latent heat release in the upper troposphere, causing the lapse rate to decrease ($\Delta\Gamma < 0$). This is known as the *lapse rate feedback.*
   (a) **(1 point)** Starting from the relation $T_s = T_e + \Gamma z_e$, derive an expression for the total change in surface temperature $\Delta T_s$ considering that both $z_e$ and $\Gamma$ can change, while $T_e$ remains fixed by the solar balance.

(b) **(1 point)** Physical models suggest that for the Earth, $d\Gamma/dT_s \approx -0.03\ \text{km}^{-1}$. Using $z_e \approx 5$ km, determine if this feedback is **positive** (amplifies warming) or **negative** (dampens warming), and calculate the corrected $\Delta T_s$ for the doubling of $CO_2$ (where $\Delta z_e = 150$ m, as in Question 3).

## Solutions

1. Consider a simplified model where the Earth's surface emits as a black body at temperature $T_s$. The atmosphere is represented by a single isothermal layer at temperature $T_a$. This layer absorbs a fraction $\epsilon$ of the infrared radiation (emissivity) and allows the remaining $(1-\epsilon)$ to escape directly to space through the so-called "atmospheric window".

   (a) **(2 points)** Write the energy balance equations for (i) the atmosphere, and (ii) the Earth-atmosphere system as a whole, assuming radiative equilibrium. Use $S_0$ for the solar constant and $A$ for the planetary albedo.

   **Balance Equations:**

   - *Atmosphere:* $\epsilon\sigma T_s^4 = 2\epsilon\sigma T_a^4 \implies \sigma T_a^4 = \frac{1}{2}\sigma T_s^4$.
     (The atmosphere emits $\epsilon\sigma T_a^4$ both upwards and downwards).
   - *System (Top of Atmosphere):* $\dfrac{S_0(1-A)}{4} = (1-\epsilon)\sigma T_s^4 + \epsilon\sigma T_a^4$.

   (b) **(2 points)** Prove that the surface temperature $T_s$ can be expressed as:

   $$T_s = T_e\left(\frac{2}{2-\epsilon}\right)^{1/4}, \tag{3}$$

   where $T_e$ is the effective emission temperature of the planet. Calculate $T_s$ given $T_e = 255$ K and $\epsilon = 0.77$.

   **Derivation:** Defining $\sigma T_e^4 = \dfrac{S_0(1-A)}{4}$, we substitute the atmospheric balance into the system balance:

   $$\sigma T_e^4 = \sigma T_s^4(1-\epsilon) + \epsilon\left(\frac{1}{2}\sigma T_s^4\right) = \sigma T_s^4\left(1-\frac{\epsilon}{2}\right). \tag{4}$$

   Rearranging for $T_s$:

   $$T_s^4 = \frac{T_e^4}{1-\epsilon/2} = T_e^4\left(\frac{2}{2-\epsilon}\right) \implies T_s = T_e\left(\frac{2}{2-\epsilon}\right)^{1/4}. \tag{5}$$

   For $T_e = 255$ K and $\epsilon = 0.77$:

   $$T_s = 255 \times \left(\frac{2}{1.23}\right)^{1/4} \approx 255 \times 1.129 \approx 288 \text{ K}. \tag{6}$$

2. **(2 points)** The increase in GHG concentrations effectively "closes" parts of the infrared spectrum. For $CO_2$, the radiative forcing ($\Delta F$, the change in net flux at the top of the atmosphere) follows a logarithmic law:

   $$\Delta F = \alpha \ln\left(\frac{C}{C_0}\right), \tag{7}$$

   where $\alpha = 5.35$ W/m$^{-2}$ and $C/C_0$ is the ratio of final to initial $CO_2$ concentration. Calculate the forcing $\Delta F$ for a doubling of $CO_2$ concentration ($C/C_0 = 2$). Using the Stefan-Boltzmann law calculate the required change in the effective emission temperature ($\Delta T_e$) to restore equilibrium after this forcing, assuming $T_e$ was initially 255 K. Use $\sigma = 5.67 \times 10^{-8}$ W·m$^{-2}$·K$^{-4}$.

   1. **Radiative Forcing:**

   $$\Delta F = 5.35 \times \ln(2) \approx 3.71 \text{ W/m}^2. \tag{8}$$

2. **Change in $T_e$:** Differentiating $F = \sigma T_e^4$ gives $\Delta F = 4\sigma T_e^3 \Delta T_e$.

$$\Delta T_e = \frac{\Delta F}{4\sigma T_e^3} = \frac{3.71}{4 \times 5.67 \times 10^{-8} \times (255)^3} \approx \frac{3.71}{3.76} \approx 0.986 \text{ K}. \tag{9}$$

3. **(2 points)** In the real atmosphere, heat transport is not solely radiative but also convective, maintaining a vertical lapse rate $\Gamma = -dT/dz \approx 6.5$ K/km. The radiation escaping to space is emitted, on average, from an effective altitude $z_e$ such that $T(z_e) = T_e$. As GHG concentrations increase, the atmosphere becomes more opaque, shifting the emission level $z_e$ upwards. Show that if $z_e$ increases by $\Delta z_e$, the surface temperature $T_s$ must increase by $\Delta T_s = \Gamma \Delta z_e$ to maintain the same $T_e$ (and thus restore radiative balance). If a doubling of $CO_2$ raises the emission level $z_e$ by 150 m, calculate the resulting increase in surface temperature $\Delta T_s$.

   1. **Relation between $T_s$ and $T_e$:** The temperature profile is $T(z) = T_s - \Gamma z$. At the emission level $z_e$, we have $T_e = T_s - \Gamma z_e$. Since $T_e$ is determined by the solar constant, it must remain constant in the new equilibrium. Thus:

$$T_s(\text{new}) = T_e + \Gamma(z_e + \Delta z_e), \tag{10}$$

$$\Delta T_s = [T_e + \Gamma z_e + \Gamma \Delta z_e] - [T_e + \Gamma z_e] = \Gamma \Delta z_e. \tag{11}$$

   2. **Numerical Calculation:** Given $\Gamma = 6.5$ K/km and $\Delta z_e = 0.15$ km:

$$\Delta T_s = 6.5 \times 0.15 = 0.975 \text{ K}. \tag{12}$$

   This value is known as the “Planck Feedback” or No-Feedback Climate Sensitivity.

4. In the previous Question, we assumed the lapse rate $\Gamma$ remains constant. However, in a warming atmosphere, increased evaporation leads to more latent heat release in the upper troposphere, causing the lapse rate to decrease ($\Delta\Gamma < 0$). This is known as the *lapse rate feedback.*

   (a) **(1 point)** Starting from the relation $T_s = T_e + \Gamma z_e$, derive an expression for the total change in surface temperature $\Delta T_s$ considering that both $z_e$ and $\Gamma$ can change, while $T_e$ remains fixed by the solar balance.

   **Total Derivative of $T_s$:** Given $T_s = T_e + \Gamma z_e$, and assuming $T_e$ is constant:

$$\Delta T_s = \frac{\partial T_s}{\partial z_e}\Delta z_e + \frac{\partial T_s}{\partial \Gamma}\Delta\Gamma, \tag{13}$$

$$\Delta T_s = \Gamma \Delta z_e + z_e \Delta\Gamma. \tag{14}$$

   (b) **(1 point)** Physical models suggest that for the Earth, $d\Gamma/dT_s \approx -0.03$ km$^{-1}$. Using $z_e \approx 5$ km, determine if this feedback is **positive** (amplifies warming) or **negative** (dampens warming), and calculate the corrected $\Delta T_s$ for the doubling of $CO_2$ (where $\Delta z_e = 150$ m, as in Question 3).

   **Feedback Analysis:** Substituting $\Delta\Gamma = \dfrac{d\Gamma}{dT_s}\Delta T_s$:

$$\Delta T_s = \Gamma \Delta z_e + z_e \left(\frac{d\Gamma}{dT_s}\right)\Delta T_s. \tag{15}$$

   Solving for $\Delta T_s$:

$$\Delta T_s \left(1 - z_e \frac{d\Gamma}{dT_s}\right) = \Gamma \Delta z_e \quad \Longrightarrow \quad \Delta T_s = \frac{\Gamma \Delta z_e}{1 - z_e \dfrac{d\Gamma}{dT_s}}. \tag{16}$$

**Numerical Calculation:** With $z_e = 5$ km and $d\Gamma/dT_s = -0.03\,\text{km}^{-1}$, we obtain $z_e\,(d\Gamma/dT_s) = 5 \times (-0.03) = -0.15$. The denominator in $\Delta T_s$, Eq. (16) becomes $1 - (-0.15) = 1.15$, and therefore:

$$\Delta T_s = \frac{0.975\ \text{K}}{1.15} \approx 0.848\ \text{K}. \tag{17}$$

**Conclusion:** Since the resulting $\Delta T_s$ (0.848 K) is smaller than the initial $\Delta T_s$ (0.975 K), the lapse rate feedback is a **negative feedback**. Physically, this is because on global average the upper trosposphere warms more than the surface, making it a more efficient radiator of energy to space.

## 4 Quantum Machine Learning

# Physics-aware quantum learning

In Quantum Machine Learning, quantum states are often generated by parameterized models (quantum ansätze) whose parameters are trained by minimizing a cost function. A common task is state preparation by energy minimization, analogous to training a model by minimizing a loss.

Consider the Hamiltonian of the harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2, \tag{1}$$

where $\omega$ is the natural frequency of the oscillator and $m$ its mass. Consider the variational state wave function

$$\psi_\sigma(x) = \langle x|\psi_\sigma\rangle = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{2}$$

where $\sigma$ is the variational parameter. Notice that the wave function is normalized. Here, the Gaussian wave packet amplitude plays the role of a trainable parameter, analogous to a weight in a classical Neural Network.

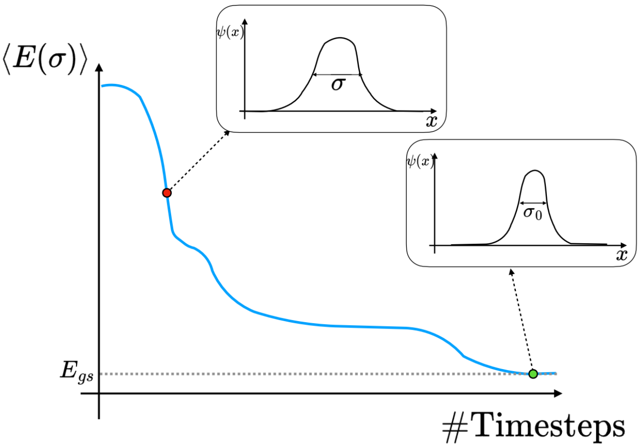

Figure 1: Sketch of the minimization process in a neural network.

In a Physics-Informed Neural Networks, a variational model is trained by minimizing a loss that penalizes violations of governing equations. In the case of Quantum Mechanics and wave packets, the loss function would be the Schrödinger equation:

$$C(\alpha) = i\hbar\partial_t\langle x|\psi_\alpha(t)\rangle - \langle x|\hat{H}|\psi_\alpha(t)\rangle. \tag{3}$$

## QUESTIONS

1. **(3 points)** Compute the cost function $E(\sigma) = \langle\psi_\sigma|\hat{H}|\psi_\sigma\rangle$.
2. **(1 point)** Find the value $\sigma_0$ that minimizes the cost function $E(\sigma)$.
3. **(2 points)** Consider a scenario where $\hat{H} = \hat{p}^2/2m$ is the free Hamiltonian, and our wave packet takes the form previously obtained with a complex amplitude,

$$\psi_\alpha(x,t) = \langle x|\psi_\alpha(t)\rangle = \frac{1}{\pi^{1/4}}\left[\frac{\sigma_0}{\sigma_0^2 + i\alpha^2(t)}\right]^{1/2} \exp\left\{-\frac{x^2}{2\left[\sigma_0^2 + i\alpha^2(t)\right]}\right\}. \tag{4}$$

   Compute the loss function $C(\alpha)$.

4. **(2 points)** Find the value of $\alpha(t)$ that solves $C(\alpha) = 0$ for the initial condition $\alpha(0) = 0$. How does the uncertainty in position $\Delta x(t)$ evolves in time?

5. **(2 points)** Consider the momentum-space wave function $\tilde{\psi}(p,t) = \langle p|\psi(t)\rangle$. Discuss how the momentum uncertainty $\Delta p(t)$ evolves in time.

## Math help:

Definite Gaussian integrals:

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}. \tag{5}$$

$$\int_{-\infty}^{\infty} u^2 e^{-u^2} du = \frac{\sqrt{\pi}}{2}. \tag{6}$$

## Solutions

1. **(3 points)** Compute the cost function $E(\sigma) = \langle \psi_\sigma | \hat{H} | \psi_\sigma \rangle$.

   The cost function is computed as follows

$$\begin{aligned}
E(\sigma) &= \int_{-\infty}^{\infty} dx \psi_\sigma^*(x) \hat{H} \psi_\sigma(x) \\
&= \int_{-\infty}^{\infty} dx \left(\frac{1}{\pi\sigma^2}\right)^{1/2} e^{-\frac{x^2}{2\sigma^2}} \left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2 x^2\right) e^{-\frac{x^2}{2\sigma^2}} \\
&= \left(\frac{1}{\pi\sigma^2}\right)^{1/2} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}} \left(-\frac{\hbar^2 x^2}{2m\sigma^4} + \frac{\hbar^2}{2m\sigma^2} + \frac{1}{2}m\omega^2 x^2\right) e^{-\frac{x^2}{2\sigma^2}} \\
&= \left(\frac{1}{\pi\sigma^2}\right)^{1/2} \left[\left(\frac{m\omega^2}{2} - \frac{\hbar^2}{2m\sigma^4}\right) \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{\sigma^2}} x^2 + \frac{\hbar^2}{2m\sigma^2} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{\sigma^2}}\right] \\
&= \left(\frac{1}{\pi\sigma^2}\right)^{1/2} \left[\left(\frac{m\omega^2}{2} - \frac{\hbar^2}{2m\sigma^4}\right) \frac{\sqrt{\pi\sigma^2}\sigma^2}{2} + \frac{\hbar^2}{2m\sigma^2}\sqrt{\pi\sigma^2}\right] \\
&= \frac{m\omega^2\sigma^2}{4} - \frac{\hbar^2}{4m\sigma^2} + \frac{\hbar^2}{2m\sigma^2} \\
&= \frac{m\omega^2\sigma^2}{4} + \frac{\hbar^2}{4m\sigma^2}.
\end{aligned} \tag{7}$$

2. **(1 point)** Find the value $\sigma_0$ that minimizes the cost function $E(\sigma)$.

   To find the minimum of $E(\sigma)$ we have to differentiate and solve

$$\frac{d}{d\sigma}E(\sigma) = \frac{\sigma m\omega^2}{2} - \frac{\hbar^2}{2m\sigma^3} = 0 \quad \Longrightarrow \quad \sigma^2 = \frac{\hbar}{m\omega}. \tag{8}$$

3. **(2 points)** Consider a scenario where $\hat{H} = \hat{p}^2/2m$ is the free Hamiltonian, and our wave packet takes the form previously obtained with a complex amplitude,

$$\psi_\alpha(x,t) = \langle x | \psi_\alpha(t) \rangle = \frac{1}{\pi^{1/4}} \left[\frac{\sigma_0}{\sigma_0^2 + i\alpha^2(t)}\right]^{1/2} \exp\left\{-\frac{x^2}{2\left[\sigma_0^2 + i\alpha^2(t)\right]}\right\}. \tag{9}$$

   Compute the loss function $C(\alpha)$.

   The cost function is computed directly by differentiation with respect to $t$ and $x$ to yield

$$C(\alpha) = -\frac{\hbar}{2m\pi^{1/4}} \frac{\left[-x^2 + \sigma_0^2 + i\alpha(t)^2\right]\left[\hbar - 2m\alpha(t)\dot{\alpha}(t)\right]}{\left[\sigma_0^2 + i\alpha(t)^2\right]^2} \sqrt{\frac{\sigma_0}{\sigma_0^2 + i\alpha(t)^2}}\, e^{-\frac{x^2}{2\sigma_0^2 + 2i\,\alpha(t)^2}}. \tag{10}$$

4. **(2 points)** Find the value of $\alpha(t)$ that solves $C(\alpha) = 0$ for the initial condition $\alpha(0) = 0$. How does the uncertainty in position $\Delta x(t)$ evolves in time?

   o solve the cost function $C(\alpha)$ for all $x$ we are forced to solve

$$\hbar - 2m\alpha(t)\,\dot{\alpha}(t) = 0, \tag{11}$$

   which can be rewritten as

$$m\frac{d}{dt}\alpha^2(t) = \hbar, \tag{12}$$

with straightforward solution

$$\alpha(t) = \sqrt{\frac{\hbar t}{m}}. \tag{13}$$

With this solution we can compute the probability amplitude of the wave function as

$$|\psi(t,x)|^2 = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{m^2 x^2 \sigma_0^2}{\hbar^2 t^2 + m^2 \sigma_0^4}\right) \sqrt{\frac{m\sigma_0}{-i\hbar t + m\sigma_0^2}} \sqrt{\frac{m\sigma_0}{i\hbar t + m\sigma_0^2}}, \tag{14}$$

which is a Gaussian with an associated standard deviation

$$\Delta x(t) = \sigma(t) = \sqrt{\frac{\hbar^2 t^2 + m^2 \sigma_0^4}{2m^2 \sigma_0^2}}. \tag{15}$$

5. **(2 points)** Consider the momentum-space wave function $\tilde{\psi}(p,t) = \langle p|\psi(t)\rangle$. Discuss how the momentum uncertainty $\Delta p(t)$ evolves in time.

   The free Hamiltonian is diagonal in momentum space,

   $$\hat{H} = \frac{p^2}{2m}, \tag{16}$$

   so that the wave function evolves as

   $$\tilde{\psi}(t,p) = e^{-ip^2 t/2m\hbar}\, \tilde{\psi}(0,p), \tag{17}$$

   while the probability density does not evolve,

   $$|\tilde{\psi}(t,p)|^2 = \tilde{\psi}^*(0,p) e^{ip^2 t/2m\hbar} \tilde{\psi}(0,p) e^{-ip^2 t/2m\hbar} = |\tilde{\psi}(0,p)|^2, \tag{18}$$

   but remains constant.

## 5 Optomechanics

# Can light hold matter? Trapping a nanoparticle with a Gaussian beam

Beyond its role as a carrier of energy and information, light can exert mechanical forces and control the motion of matter. This remarkable property underlies optical trapping, a technique pioneered by Arthur Ashkin and recognized with the Nobel Prize in Physics in 2018[2]. In this problem, you will analyze the optical forces exerted by a focused laser beam on a dielectric nanoparticle in vacuum. The particle is assumed to be much smaller than the wavelength and can therefore be modeled as an induced electric dipole with complex polarizability

$$\alpha = \alpha' + i\alpha''. \tag{1}$$

Consider a linearly polarized laser beam propagating along the z-axis, with a non-uniform transverse intensity profile. The complex electric field of the beam is given by

$$\mathbf{E}(x,z) = E_0 \, \frac{w_0}{w(z)} \, e^{-x^2/w_0^2 + ikz} \hat{\mathbf{x}}, \tag{2}$$

where

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \tag{3}$$

is the beam waist radius, $w_0$ is the minimum waist, $z_R = \pi w_0^2/\lambda$ is the so-called Rayleigh range, $\lambda$ is the wavelength, and $k = 2\pi/\lambda$ is the wavenumber.

## QUESTIONS

1. **(5 points)** Calculate the time-averaged optical force acting on a dipolar particle can be written as the sum of a gradient force and a scattering (radiation pressure) force:

$$\langle \mathbf{F} \rangle = \langle \mathbf{F}_{grad} \rangle + \langle \mathbf{F}_{rad} \rangle = \alpha' \nabla \langle W_E \rangle + \alpha'' \, \frac{k}{c} \, \langle \mathbf{S} \rangle, \tag{4}$$

where $\mathbf{S}$ is the Poynting vector and $W_E$ is the electromagnetic energy density. Express the total force $\langle \mathbf{F} \rangle$ explicitly in terms of $\alpha'$, $\alpha''$, the peak intensity

$$I_0 = \frac{1}{2\eta} |E_0|^2, \tag{5}$$

where $\eta = 1/c\epsilon_0$ being the impedance of free space, and the spatial coordinates $x$ and $z$.

2. **(3 points)** Assuming that the radiation force is negligible compared to the gradient force, determine the equilibrium position of the trapped particle.

3. **(2 points)** Compute the trap stiffness (spring constants) $\kappa_x$ and $\kappa_z$ around equilibrium, defined through the linear restoring force (previously calculated) that can be written as:

$$\mathbf{F} \simeq -\kappa_x x \hat{\mathbf{x}} - \kappa_z z \hat{\mathbf{z}}. \tag{6}$$

Assume again that the radiation force is negligible compared to the gradient force.

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[2] The Nobel Prize in Physics 2018; Arthur Ashkin – Facts – 2018

## Solutions

1. **(5 points)** Calculate the time-averaged optical force acting on a dipolar particle can be written as the sum of a gradient force and a scattering (radiation pressure) force:

$$\langle \mathbf{F} \rangle = \langle \mathbf{F}_{grad} \rangle + \langle \mathbf{F}_{rad} \rangle = \alpha' \nabla \langle W_E \rangle + \alpha'' \, \frac{k}{c} \, \langle \mathbf{S} \rangle, \tag{7}$$

where $\mathbf{S}$ is the Poynting vector and $W_E$ is the electromagnetic energy density. Express the total force $\langle \mathbf{F} \rangle$ explicitly in terms of $\alpha'$, $\alpha''$, the peak intensity

$$I_0 = \frac{1}{2\eta} \, |E_0|^2, \tag{8}$$

where $\eta = 1/c\epsilon_0$ being the impedance of free space, and the spatial coordinates $x$ and $z$.

The total time-averaged optical force on a dipolar particle is:

$$\langle \mathbf{F} \rangle = \alpha' \nabla \langle W_E \rangle + \alpha'' \frac{k}{c} \langle \mathbf{S} \rangle, \tag{9}$$

as provided in the problem.

**Step 1:** Define the fields that will be used in the problem. The electric field, $\mathbf{E}$, was provided, but for the calculation of the Poynting vector, we also need the magnetic field $\mathbf{H}$. Thus, the electric field is given by:

$$\mathbf{E}(x,z) = E_0 \, \frac{w_0}{w(z)} \, e^{-x^2/w_0^2 - ikz} \, \hat{\mathbf{x}}, \tag{10}$$

and the magnetic field is:

$$\mathbf{H} = \frac{1}{\eta} \, \mathbf{k} \times \mathbf{E} = \frac{1}{\eta} \, E_0 \frac{w_0}{w(z)} \, e^{-x^2/w_0^2 - ikz} \, \hat{\mathbf{y}}. \tag{11}$$

**Step 2:** Calculate the energy density and Poynting vector. The electric energy density is:

$$\langle W_E \rangle = \frac{1}{4} \, \epsilon_0 |\mathbf{E}|^2 = \frac{1}{4} \, \epsilon_0 |E_0|^2 \, \frac{w_0^2}{w^2(z)} \, e^{-2x^2/w_0^2}. \tag{12}$$

Using $I_0 = \dfrac{c\epsilon_0}{2} |E_0|^2$, we can rewrite it as

$$\langle W_E \rangle = \frac{I_0}{2c} \frac{w_0^2}{w^2(z)} \, e^{-2x^2/w_0^2}. \tag{13}$$

The time-averaged Poynting vector is:

$$\langle \mathbf{S} \rangle = \frac{1}{2} \, \mathrm{Re} \left\{ \mathbf{E} \times \mathbf{H}^* \right\} = \frac{1}{2} \, \mathrm{Re} \left\{ \frac{E_0 E_0^*}{\eta} \, \frac{w_0^2}{w^2(z)} \, \left( e^{-x^2/w_0^2} \right)^2 \, e^{-ikz} e^{ikz} \right\} \hat{\mathbf{z}}. \tag{14}$$

Using $I_0 = \dfrac{c\epsilon_0}{2} |E_0|^2 = \dfrac{1}{2\eta} |E_0|^2$,

$$\langle \mathbf{S} \rangle = I_0 \, \frac{w_0^2}{w^2(z)} \, e^{-2x^2/w_0^2} \, \hat{\mathbf{z}}. \tag{15}$$

**Step 3:** Now, we calculate the gradients to retrieve the total force. First, we calculate the gradient force related to the energy density as:

$$\begin{aligned}\mathbf{F}_{grad} &= \alpha' \nabla \langle W_E \rangle = \alpha' \, \frac{I_0}{2c} \nabla \left\langle \frac{w_0^2}{w^2(z)} \, e^{-2x^2/w_0^2} \right\rangle \\ &= \alpha' \, \frac{I_0}{2c} \left( \hat{\mathbf{x}} \, \frac{\partial}{\partial x} + \hat{\mathbf{y}} \, \frac{\partial}{\partial y} + \hat{\mathbf{z}} \, \frac{\partial}{\partial z} \right) \left\{ \left[ 1 + \left( \frac{z}{z_R} \right)^2 \right]^{-1} e^{-2x^2/w_0^2} \right\}. \end{aligned} \tag{16}$$

Then,

$$\mathbf{F}_{grad} = -\alpha' \, \frac{I_0}{c} \frac{2x}{w^2(z)} \, e^{-2x^2/w_0^2} \hat{\mathbf{x}} - \alpha' \, \frac{I_0}{c} \frac{z}{z_R^2} \frac{w_0^4}{w^4(z)} \, e^{-2x^2/w_0^2} \hat{\mathbf{z}}. \tag{17}$$

On the other side, for the radiation pressure/scattering force:

$$\mathbf{F}_{rad} = \alpha'' \, \frac{k}{c} \, \langle \mathbf{S} \rangle = \alpha'' \, \frac{k}{c} \, I_0 \, \frac{w_0^2}{w^2(z)} \, e^{-2x^2/w_0^2} \, \hat{\mathbf{z}}. \tag{18}$$

Thus, the total force is:

$$\begin{aligned}\langle \mathbf{F} \rangle &= \langle \mathbf{F}_{grad} \rangle + \langle \mathbf{F}_{rad} \rangle = \alpha' \nabla \langle W_E \rangle + \alpha'' \, \frac{k}{c} \langle \mathbf{S} \rangle \\ &= -\alpha' \, \frac{I_0}{c} \frac{2x}{w^2(z)} \, e^{-2x^2/w_0^2} \hat{\mathbf{x}} - \alpha' \, \frac{I_0}{c} \frac{z}{z_R^2} \frac{w_0^4}{w^4(z)} \, e^{-2x^2/w_0^2} \hat{\mathbf{z}} + \alpha'' \, \frac{k}{c} \, I_0 \frac{w_0^2}{w^2(z)} \, e^{-2x^2/w_0^2} \, \hat{\mathbf{z}}. \end{aligned} \tag{19}$$

In terms of componentes, the total explicit force is:

$$F_x = -\alpha' \, \frac{I_0}{c} \frac{2x}{w^2(z)} \, e^{-2x^2/w_0^2}, \tag{20}$$

$$F_z = -\alpha' \, \frac{I_0}{c} \frac{2z}{z_R^2} \frac{w_0^4}{w^4(z)} \, e^{-2x^2/w_0^2} + \alpha'' \, \frac{k}{c} I_0 \frac{w_0^2}{w^2(z)} \, e^{-2x^2/w_0^2}. \tag{21}$$

2. **(3 points)** Assuming that the radiation force is negligible compared to the gradient force, determine the equilibrium position of the trapped particle.

   Assuming $\langle \mathbf{F}_{\text{rad}} \rangle$ is negligible, $\langle \mathbf{F}_{\text{rad}} \rangle \ll \langle \mathbf{F}_{\text{grad}} \rangle$:

   $$\langle \mathbf{F} \rangle \approx \langle \mathbf{F}_{grad} \rangle = -\alpha' \, \frac{I_0}{c} \frac{2x}{w^2(z)} \, e^{-2x^2/w_0^2} \hat{\mathbf{x}} - \alpha' \, \frac{I_0}{c} \frac{z}{z_R^2} \frac{w_0^4}{w^4(z)} \, e^{-2x^2/w_0^2} \hat{\mathbf{z}}. \tag{22}$$

   In the equilibrium position, $\langle \mathbf{F}_{\text{grad}} \rangle \simeq \mathbf{0}$:

   - $F_x = 0 \implies -\alpha' \, \frac{I_0}{c} \, e^{-2x^2/w^2} \frac{2x}{w^2} = 0$, which gives $x = 0$.
   - $F_y = 0$, as there is no $y$ dependence.
   - $F_z = 0 \implies z = 0$ (at the focus, where the gradient vanishes).

   The equilibrium position is then $x = 0, z = 0, \forall y \in \mathbb{R}$.

3. **(2 points)** Compute the trap stiffness (spring constants) $\kappa_x$ and $\kappa_z$ around equilibrium, defined through the linear restoring force (previously calculated) that can be written as:

$$\mathbf{F} \simeq -\kappa_x x \hat{\mathbf{x}} - \kappa_z z \hat{\mathbf{z}}. \tag{23}$$

Assume again that the radiation force is negligible compared to the gradient force.

The force near equilibrium is $\mathbf{F} = -k_x \hat{\mathbf{x}} - k_z \hat{\mathbf{z}} \approx -\kappa_x x\, \hat{\mathbf{x}} - \kappa_z z\, \hat{\mathbf{z}}$. Evaluating the expressions obtained before:

- **Transverse Stiffness**:

$$\kappa_x = -\alpha' \frac{I_0}{c} \frac{2x}{w^2(z)}\, e^{-2x^2/w_0^2}. \tag{24}$$

- **Axial Stiffness**:

$$\kappa_z = -\alpha' \frac{I_0}{c z_R^2} \frac{w_0^4}{w^4(z)}\, e^{-2x^2/w_0^2}. \tag{25}$$

## 6 Quantum Optics

# A photon's Shakespearean dilemma – $|\text{to be}\rangle$ or $|\text{not to be}\rangle$ . . . interference!

Consider an ideal Mach–Zehnder interferometer consisting of two identical 50:50 beamsplitters, two perfect mirrors, and two ideal single-photon detectors $D_1$ and $D_2$, as shown below. To investigate the effect of the imbalance produced by a different path length, the interferometer is arranged in such a way that arm 2 is slightly larger than arm 1. This results in a path-length difference $\Delta L = L_2 - L_1 = 150$ nm between the two possible photon pathways.

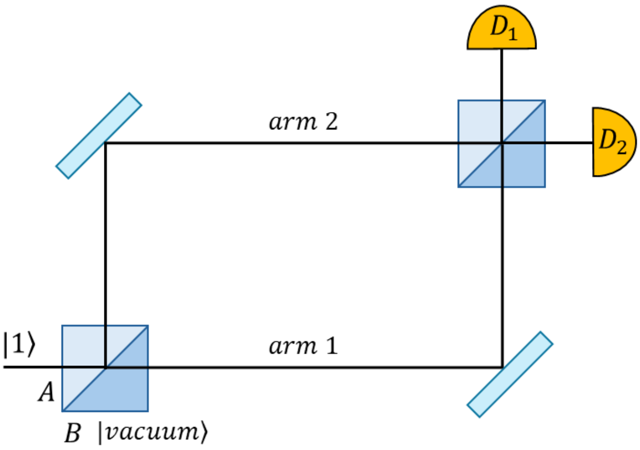

Figure 1: Two identical masses connected by three springs.

A single monochromatic photon of wavelength $\lambda = 600$ nm enters the interferometer through its input port $A$, while input port $B$ is left in the vacuum state. Each beamsplitter acts on an incoming photon as follows:

$$|1\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|1\rangle_{\mathrm{t}} + i\,|1\rangle_{\mathrm{r}}\right), \tag{1}$$

where $t$ and $r$ stand, respectively, for transmission and reflection at the beamsplitter; $i$ specifies the $\pi/2$-phase shift upon reflection at the beamsplitter.

## QUESTIONS

1. **(2 points) State evolution.** Write the expression of the quantum state of the photon: (i) immediately after the first beamsplitter, and (ii) right before the second beamsplitter. In case (ii), leave the expression in terms of the relative phase $\phi$ acquired due to the path-length difference $\Delta L$. Compute the value of $\phi$ with the numerical data provided.

2. **(3 points) Detection probabilities.** Determine the photon quantum state behind the second beamsplitter, and obtain the analytical expression of the detection probabilities $P_{D_1}$ and $P_{D_2}$ as functions of $\phi$. Evaluate numerically both probabilities with the numerical data provided.

3. **(3 points) Spectral coherence.** Assume that the photons used in the experiment have a Gaussian spectral distribution,

$$g(\lambda) = \frac{1}{\sqrt{\pi}\,\Delta\lambda}\,\exp\left[-\left(\frac{\lambda-\lambda_0}{\Delta\lambda}\right)^2\right], \tag{2}$$

with $\lambda_0 = 600$ nm, $\Delta\lambda = 10$ nm, and satisfying $\int g(\lambda)d\lambda = 1$. Write the integral expression for the spectrally averaged probability $\langle P_{D_1}\rangle$, and show that the interference visibility at $D_1$ is reduced by a factor

$$\mathcal{V} = \exp\left[-\left(\frac{\pi\,\Delta L}{\ell_c}\right)^2\right], \tag{3}$$

where $\ell_c \equiv \lambda_0^2/\Delta\lambda$ is the photon coherence length. Compute the numerical value of $\mathcal{V}$ and comment on the relevance of the phase shift in the interference visibility at $D_1$.

**Hint.** To proceed with the integral, note that, because $\Delta\lambda \ll \lambda_0$, Eq. (2) is sharply peaked around $\lambda \approx \lambda_0$. Hence, $g(\lambda)$ is essentially nonzero in a narrow neighborhood of $\lambda = \lambda_0$, quickly vanishing everywhere else. This allows us to consider the following reasonable approximation:

$$\frac{\lambda\lambda_0}{\lambda-\lambda_0} \approx \frac{\lambda_0^2}{\lambda-\lambda_0}. \tag{4}$$

4. **(2 points) Quantum-to-classical optics limit.** The experiment is repeated with $N \gg 1$ independent photons. Obtain the mean and the variance of the number of photons (counts) detected by $D_1$. Determine the trend of the ratio $\Delta n/\langle n\rangle$ in the limit $N \to \infty$, where $\Delta n = \mathrm{Var}(n_{D_1})$ is the standard deviation of the quantum noise fluctuations, and discuss it in relation to the relevance of quantum fluctuations in such a limit.

## Solutions

1. **(2 points) State evolution.** Write the expression of the quantum state of the photon: (i) immediately after the first beamsplitter, and (ii) right before the second beamsplitter. In case (ii), leave the expression in terms of the relative phase $\phi$ acquired due to the path-length difference $\Delta L$. Compute the value of $\phi$ with the numerical data provided.

   From Fig. 1, we readily find that the photon state after the first beamsplitter is

   $$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |1\rangle_1 + i\,|1\rangle_2 \right), \tag{5}$$

   where 1 and 2 denote the possible pathway (arm) followed by the photon. This state represents an unbiased superposition of both pathways.

   Because of the path-length difference between the two arms, the photon acquires a phase along each arm, such that right before the second beamsplitter its quantum state reads as

   $$|\psi_\phi\rangle = \frac{1}{\sqrt{2}} \Big( e^{ikL_1}\,|1\rangle_1 + i e^{ikL_2}\,|1\rangle_2 \Big), \tag{6}$$

   which can be recast in a more compact form, in terms of the relative phase $\phi = k\Delta L = k(L_2 - L_1)$ (and assigning zero phase to arm 1), as

   $$|\psi_\phi\rangle = \frac{1}{\sqrt{2}} \Big( |1\rangle_1 + i e^{i\phi}\,|1\rangle_2 \Big). \tag{7}$$

   This is the quantum state entering the second beamsplitter.

   The explicit value of the relative phase $\phi$ is obtained after substitution of the specific numerical data provided:

   $$\phi = k\Delta L = \frac{2\pi}{\lambda}\,\Delta L = \frac{2\pi}{600\ \text{nm}} \times 150\ \text{nm} = \frac{\pi}{2}. \tag{8}$$

2. **(3 points) Detection probabilities.** Determine the photon quantum state behind the second beamsplitter, and obtain the analytical expression of the detection probabilities $P_{D_1}$ and $P_{D_2}$ as functions of $\phi$. Evaluate numerically both probabilities with the numerical data provided.

   The second beamsplitter acts in the same way as the first one, although now, first, we have to consider its action on each state separately:

   $$|1\rangle_1 \to \frac{1}{\sqrt{2}} \left( |1\rangle_{D_1} + i|1\rangle_{D_2} \right), \quad |1\rangle_2 \to \frac{1}{\sqrt{2}} \left( i|1\rangle_{D_1} + |1\rangle_{D_2} \right). \tag{9}$$

   Making the corresponding substitutions, we obtain the quantum state of the photon behind the second beamsplitter,

   $$|\psi_{\rm out}\rangle = \frac{1}{2} \Big[ (1 - e^{i\phi})\,|1\rangle_{D_1} + i(1 + e^{i\phi})\,|1\rangle_{D_2} \Big], \tag{10}$$

   where subscripts $D_1$ and $D_2$ indicate which detector will be reached by each component. The corresponding prefactors give the respective detection probabilities:

   $$\begin{aligned} \text{Detector } D_1\text{:} \quad P_{D_1} &= \frac{1}{4}|1 - e^{i\phi}|^2 = \frac{1}{2}(1 - \cos\phi), \\ \text{Detector } D_2\text{:} \quad P_{D_1} &= \frac{1}{4}|1 + e^{i\phi}|^2 = \frac{1}{2}(1 + \cos\phi). \end{aligned} \tag{11}$$

Note that, as it should be expected, regardless of $\phi$ we have:

$$P_{D_1} + P_{D_2} = 1. \tag{12}$$

When the detection probabilities are evaluated with the numerical data of the problem (i.e., for $\phi = \pi/2$), they become

$$P_{D_1} = P_{D_2} = \frac{1}{2}, \tag{13}$$

which physically means that the photon has exactly the same probability to reach any of the two detectors.

3. **(3 points) Spectral coherence.** Assume that the photons used in the experiment have a Gaussian spectral distribution,

$$g(\lambda) = \frac{1}{\sqrt{\pi}\,\Delta\lambda}\,\exp\left[-\left(\frac{\lambda-\lambda_0}{\Delta\lambda}\right)^2\right], \tag{14}$$

with $\lambda_0 = 600$ nm, $\Delta\lambda = 10$ nm, and satisfying $\int g(\lambda)d\lambda = 1$. Write the integral expression for the spectrally averaged probability $\langle P_{D_1}\rangle$, and show that the interference visibility at $D_1$ is reduced by a factor

$$\mathcal{V} = \exp\left[-\left(\frac{\pi\,\Delta L}{\ell_c}\right)^2\right], \tag{15}$$

where $\ell_c \equiv \lambda_0^2/\Delta\lambda$ is the photon coherence length. Compute the numerical value of $\mathcal{V}$ and comment on the relevance of the phase shift in the interference visibility at $D_1$.

**Hint.** To proceed with the integral, note that, because $\Delta\lambda \ll \lambda_0$, Eq. (14) is sharply peaked around $\lambda \approx \lambda_0$. Hence, $g(\lambda)$ is essentially nonzero in a narrow neighborhood of $\lambda = \lambda_0$, quickly vanishing everywhere else. This allows us to consider the following reasonable approximation:

$$\frac{\lambda\lambda_0}{\lambda-\lambda_0} \approx \frac{\lambda_0^2}{\lambda-\lambda_0}. \tag{16}$$

The spectral average of $P_{D_1}$ over the Gaussian distribution $g(\lambda)$ is given by the integral

$$\langle P_{D_1}\rangle = \int g(\lambda)P_{D_1}(\lambda)d\lambda, \tag{17}$$

where

$$P_{D_1}(\lambda) = \frac{1}{2}\left[1-\cos\left(\frac{2\pi\Delta L}{\lambda}\right)\right], \tag{18}$$

which depends on the wavelength $\lambda$, because the relative phase is also a function of $\lambda$. Since $g(\lambda)$ is normalized, the first term in Eq. (18) remains unaffected. Concerning the second term, we have

$$\langle\cos\phi\rangle = \frac{1}{\sqrt{\pi}\,\Delta\lambda}\int \exp\left[-\left(\frac{\lambda-\lambda_0}{\Delta\lambda}\right)^2\right]\cos\left(\frac{2\pi\Delta L}{\lambda}\right)d\lambda. \tag{19}$$

This integral cannot be solved analytically unless we make use of the approximation provided in the statement of the problem. More specifically, if we rewrite that approximated expression as

$$\frac{1}{\lambda} \approx \frac{1}{\lambda_0} - \frac{\lambda-\lambda_0}{\lambda_0^2}, \tag{20}$$

the above integral reads as

$$\langle \cos\phi \rangle \approx \frac{1}{\sqrt{\pi}\,\Delta\lambda}\int \exp\left[-\left(\frac{\lambda-\lambda_0}{\Delta\lambda}\right)^2\right]\cos\left[\frac{2\pi\Delta L}{\lambda_0}-\frac{2\pi\Delta L}{\lambda_0^2}\,(\lambda-\lambda_0)\right]d\lambda, \tag{21}$$

which is the real part of the Gaussian integral

$$\mathcal{I}_G = \int \exp\left[-\left(\frac{\lambda-\lambda_0}{\Delta\lambda}\right)^2\right]\exp\left[\frac{i2\pi\Delta L}{\lambda_0}-\frac{i2\pi\Delta L}{\lambda_0^2}\,(\lambda-\lambda_0)\right]d\lambda. \tag{22}$$

Given that $\Delta\lambda \ll \lambda_0$, the spectral distribution is highly localized and therefore we can extend the integration limits, as

$$\mathcal{I}_G \approx \int_{-\infty}^{\infty} \exp\left[-\left(\frac{\lambda-\lambda_0}{\Delta\lambda}\right)^2\right]\exp\left[\frac{i2\pi\Delta L}{\lambda_0}-\frac{i2\pi\Delta L}{\lambda_0^2}\,(\lambda-\lambda_0)\right]d\lambda. \tag{23}$$

Rearranging terms, we can recast this integral in the form of the Gaussian integral shown in the statement, so that we finally obtain

$$\mathcal{I}_G \approx \sqrt{\pi}\;\Delta\lambda\;\exp\left[-\left(\frac{\pi\Delta L\Delta\lambda}{\lambda_0^2}\right)^2\right]\exp\left(\frac{2\pi\Delta L}{\lambda_0}\right), \tag{24}$$

from which we find that the spectrally averaged probability requested reads as

$$\langle P_{D_1}\rangle = \frac{1}{2}\left[1-\mathcal{V}\cos\left(\frac{2\pi\Delta L}{\lambda_0}\right)\right], \tag{25}$$

where

$$\mathcal{V} = \exp\left[-\left(\frac{\pi\Delta L\Delta\lambda}{\lambda_0^2}\right)^2\right] \tag{26}$$

is the visibility factor. Numerical evaluation of $\mathcal{V}$ with the data provided renders

$$\mathcal{V} \approx \exp(-1.7\times 10^{-4}) \approx 0.9998, \tag{27}$$

which indicates negligible loss of coherence. Time coherence is enough here to counter-balance the unbalance produced by the path-length difference, thus almost fully preserving interference fringes.

4. **(2 points) Quantum-to-classical optics limit.** The experiment is repeated with $N \gg 1$ independent photons. Obtain the mean and the variance of the number of photons (counts) detected by $D_1$. Determine the trend of the ratio $\Delta n/\langle n\rangle$ in the limit $N\to\infty$, where $\Delta n = \mathrm{Var}(n_{D_1})$ is the standard deviation of the quantum noise fluctuations, and discuss it in relation to the relevance of quantum fluctuations in such a limit.

For $N$ independent photons, the average photon number registered by $D_1$ is readily obtained by multiplying the number of photons detected by the probability $P_{D_1}$:

$$\langle n_{D_1}\rangle = NP_{D_1} \tag{28}$$

which coincides with the classical intensities (normalized to $N$, which is proportional, in turn, to the incident intensity $I_0$ of the light beam entering the interferometer). On the other hand, since each detection is a binomial process (either there is a photon detection or not), the variance is obtained as

$$\mathrm{Var}(n_{D_1}) = NP_{D_1}(1-P_{D_1}). \tag{29}$$

Because the standard deviation grows as $\sqrt{N}$, while the average detection signal does it as $N$, we find that

$$\frac{\Delta n}{\langle n \rangle} \sim \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} \to 0 \quad (N \to \infty). \tag{30}$$

This result implies that quantum fluctuations disappear in the limit $N \gg 1$, the behavior becomes deterministic, and we recover the limit of classical optics. In other words, classical light corresponds to the high photon number described by coherent states, while the quantum nature of photons manifests under low number conditions.

## 7 Thermodynamics & Quantum Information

# Maxwell's demon goes quantum

We consider an ideal gas contained in a cylinder of fixed total volume $V$, undergoing a thermodynamic cycle. The gas has an internal quantum degree of freedom that is decoupled from the rest.

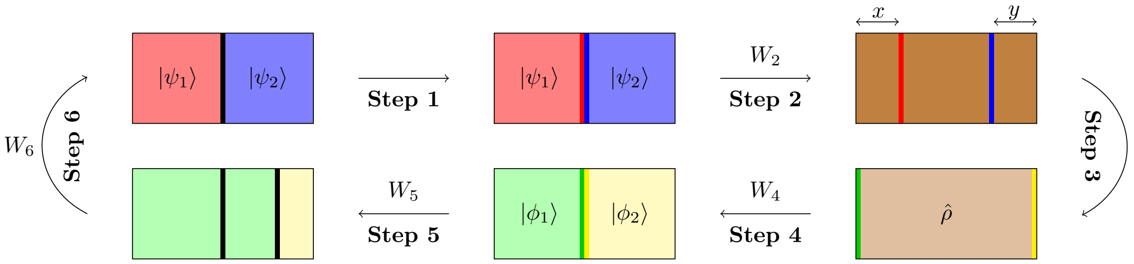

Figure 1: Thermodynamic cycle performed by the gas.

The cycle, depicted in Fig. 1, starts off with a gas divided in two portions of $N/2$ particles whose internal degrees of freedom are in two (possibly *non-orthogonal*) states $|\psi_1\rangle$ (left, in red) and $|\psi_2\rangle$ (right, in blue). Each gas occupies half of the total volume available, and is separated by a fixed opaque wall, as represented in the top left picture of Fig. 1. The cycle can be divided into six steps:

- **Step 1**: The opaque wall separating the two gases is replaced by two "membranes", one red and one blue, initially at the centre of the cylinder, the red one to the left of the blue one (see Fig. 1). These special membranes operate as follows: the red one lets the particles whose internal degree of freedom is in state $|\psi_1\rangle$ go through, but bounces back the ones whose internal degree of freedom is in state $|\psi_2\rangle$. That is, it is *transparent* to the former, and *opaque* to the latter. The blue membrane works identically, but being opaque to the particles in state $|\psi_1\rangle$, and transparent to the ones in state $|\psi_2\rangle$. However, the membranes do not operate perfectly. Instead, when they encounter a gas particle, the probability that the membranes correctly identify the quantum state of their internal degree of freedom is Prob('correct identification') $= (1+\delta)/2$, where $\delta \in [0,1]$, with $\delta = 1$ corresponding to perfect discrimination between $|\psi_1\rangle$ and $|\psi_2\rangle$, and $\delta = 0$ corresponding to random guessing. We make the simplifying assumptions that 1) the state of the particles is not affected by the interaction with the membranes, and 2) once a particle is identified as being in a particular state, that is how it is identified by *both* membranes in any future collision.

- **Step 2**: The membranes are released and allowed to move without friction.

- **Step 3**: After they have come to a halt, the membranes are removed, and the gas is allowed to mix freely, yielding a mixture whose internal quantum degrees of freedom are collectively in a mixed state $\hat{\rho} = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$, which can be diagonalized as $\hat{\rho} = c\,|\phi_1\rangle\langle\phi_1| + (1-c)\,|\phi_2\rangle\langle\phi_2|$, for some orthogonal states $|\phi_1\rangle$ and $|\phi_2\rangle$, and some real constant $c \in [0,1]$.

- **Step 4**: Because $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal to each other, they are perfectly distinguishable, and therefore we can now introduce two ideal "filters", such that one filter is totally transparent to $|\phi_1\rangle$ and totally opaque to $|\phi_2\rangle$ (the green one, in Fig. 1), while the other filter is totally opaque to $|\phi_1\rangle$ and totally transparent to $|\phi_2\rangle$ (the yellow one, in Fig. 1). Each filter is introduced at one different end of the cylinder to compress isothermally the portion of the gas they are opaque to, until they meet at the centre of the cylinder.

- **Step 5**: The filters are replaced and exchanged by an opaque wall. The wall is released and allowed to move isothermally and infinitesimally slow, so that the work performed by the most abundant state (e.g., $|\phi_1\rangle$ in Fig. 1) is partly used to compress the less abundant one, and the rest is extracted.
- **Step 6**: Another opaque wall is introduced in the middle of the cylinder, dividing it into three sectors. In each of these sectors, a unitary is applied to bring the internal quantum degree of freedom of the gas particles in that sector to the state of the gas occupying that sector at the beginning of the cycle (e.g., in Fig. 1, the leftmost sector is taken to the state $|\psi_1\rangle$, and the other two are taken to $|\psi_2\rangle$). We assume this operation can be performed infinitely slowly and that no work is necessary, i.e., $W_6 = 0$.

## QUESTIONS

1. **(3 points)** In **Step 2**, assuming that the process is isothermal, compute the distances $x$ and $y$ that separate the red and blue membranes from the walls of the cylinder once they stop moving completely, as well as the work $W_2$ performed by the gas in the process (see Fig. 1).
2. **(1 point)** In **Step 4**, compute the work $W_4$ necessary to carry out this step.
3. **(2 points)** In **Step 5**, compute the work $W_5$ extracted in this step. If possible, write the result in terms of the von Neumann entropy of $\hat{\rho}$, $S(\hat{\rho}) := -\mathrm{Tr}(\hat{\rho}\log_2 \hat{\rho})$.
4. **(2 points)** From the computation of the total work extracted in the full cycle, determine the implicit inequality that must be fulfilled by $\delta$ in order to satisfy the second law of thermodynamics.
5. **(2 points)** Using the result from Question 4, deduce that the second law of thermodynamics implies that non-orthogonal states cannot be perfectly distinguishable.

## Solutions

The problem is mostly based on Phys. Rev. E **109**, 014119 (2024), where more details about the cycle and its assumptions can be found. Throughout the solution, the sign criterion is chosen such that work is considered negative if its extracted from the system.

1. **(3 points)** In **Step 2**, assuming that the process is isothermal, compute the distances $x$ and $y$ that separate the red and blue membranes from the walls of the cylinder once they stop moving completely, as well as the work $W_2$ performed by the gas in the process (see Fig. 1).

   During Step 2, a fraction $(1-\delta)/2$ of the particles that were in state $|\psi_1\rangle$ are mistakenly identified as being in state $|\psi_2\rangle$. These particles now exert pressure on the red wall, and are constrained to move between it and the leftmost end of the cylinder. Similarly, a fraction $(1-\delta)/2$ of the particles that were in state $|\psi_2\rangle$ are mistakenly identified as being in state $|\psi_1\rangle$, and they exert pressure on the blue wall, constrained to move between it and the rightmost end of the cylinder. Thus, the red wall feels pressure a) on its left, from those particles in state $|\psi_1\rangle$ that were identified as being in state $|\psi_2\rangle$, and b) on its right, from those particles that were correctly identified as being in state $|\psi_2\rangle$, which move freely between the red wall and the rightmost end of the cylinder. Since the red wall is transparent to the rest of particles, it stops moving whenever these two pressures equalize. Let $x$ be the fraction of the total volume to the left of the red wall once it reaches equilibrium, then

   $$\frac{1-\delta}{2}\,\frac{NT}{2xV} = \frac{1+\delta}{2}\,\frac{NT}{2(1-x)V} \;\Rightarrow\; x = \frac{1-\delta}{2}. \tag{1}$$

   The situation between the red and the blue wall is symmetric under the swap $1 \leftrightarrow 2$, and therefore $x = y$. The work involved in these expansions is then

   $$\begin{aligned} W_2 &= -Nk_{\mathrm{B}}T\left(\frac{1-\delta}{4}\int_{V/2}^{xV}\frac{\mathrm{d}v}{v} - \frac{1+\delta}{4}\int_{V/2}^{(1-x)V}\frac{\mathrm{d}v}{v} - \frac{1-\delta}{4}\int_{V/2}^{yV}\frac{\mathrm{d}v}{v} - \frac{1+\delta}{4}\int_{V/2}^{(1-y)V}\frac{\mathrm{d}v}{v}\right) \\ &= -Nk_{\mathrm{B}}T\left(1 + \frac{1-\delta}{2}\log_2\frac{1-\delta}{2} + \frac{1+\delta}{2}\log_2\frac{1+\delta}{2}\right)\ln 2 \\ &= -Nk_{\mathrm{B}}T\left[1 - H\left(\frac{1+\delta}{2}\right)\right]\ln 2, \end{aligned} \tag{2}$$

   where $H(p) := -p\log_2 p - (1-p)\log_2(1-p)$ is the Shannon entropy of the binary distribution with probability $p$.

2. **(1 point)** In **Step 4**, compute the work $W_4$ necessary to carry out this step.

   Each portion of the gas goes from occupying the full volume of the cylinder to half of it, and therefore the work involved in this process is

   $$W_4 = -[cN + (1-c)N]k_{\mathrm{B}}T\int_{V}^{V/2}\frac{\mathrm{d}v}{v} = Nk_{\mathrm{B}}T\ln 2. \tag{3}$$

3. **(2 points)** In **Step 5**, compute the work $W_5$ extracted in this step. If possible, write the result in terms of the von Neumann entropy of $\hat{\rho}$, $S(\hat{\rho}) := -\mathrm{Tr}(\hat{\rho}\,\log_2\hat{\rho})$.

   The equilibrium between both sides of the wall is reached when their pressures are equal. Since both sides only differ by their quantities, at the end of the process both need to occupy volumes proportional to their quantities, namely, $cV$ and $(1-c)V$. The work involved in this compression-expansion process is

$$\begin{aligned} W_5 &= -cNk_{\mathrm{B}}T\int_{V/2}^{cV}\frac{\mathrm{d}v}{v} - (1-c)Nk_{\mathrm{B}}T\int_{V/2}^{(1-c)V}\frac{\mathrm{d}v}{v} = Nk_{\mathrm{B}}T[-c\ln c - (1-c)\ln(1-c) - 1] \\ &= -Nk_{\mathrm{B}}T[1 - S(\hat{\rho})]\ln 2, \end{aligned} \tag{4}$$

where in the last step we have used that

$$-c\log_2 c - (1-c)\log_2(1-c) = H(c) = S(\hat{\rho}). \tag{5}$$

4. **(2 points)** From the computation of the total work extracted in the full cycle, determine the implicit inequality that must be fulfilled by $\delta$ in order to satisfy the second law of thermodynamics.

   The net work received by the system during the cycle is

   $$W_t = W_2 + W_4 + W_5 = -Nk_{\mathrm{B}}T\left[1 - H\left(\frac{1+\delta}{2}\right) - S(\hat{\rho})\right]\ln 2. \tag{6}$$

   In order to satisfy the second law of thermodynamics, we must have $W_t \geq 0$, which implies

   $$1 - H\left(\frac{1+\delta}{2}\right) - S(\hat{\rho}) \leq 0. \tag{7}$$

5. **(2 points)** Using the result from Question 4, deduce that the second law of thermodynamics implies that non-orthogonal states cannot be perfectly distinguishable.

   If the states were perfectly distinguishable, then we could in principle set $\delta = 1$. Since $H(1) = 0$, from the previous inequality we get $S(\hat{\rho}) \geq 1$. Because $S(\hat{\rho}) = H(c) \leq 1$, we can only fulfill this inequality with $H(c) = 1$, which implies $c = 1/2$, and therefore that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal. Indeed, let us assume $c = 1/2$, then we would have

   $$\langle\psi_1|\,\hat{\rho}\,|\psi_1\rangle = \frac{1}{2}\big(|\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_1|\psi_2\rangle|^2\big) = \frac{1}{2}\big(|\langle\psi_1|\phi_1\rangle|^2 + |\langle\psi_1|\phi_2\rangle|^2\big). \tag{8}$$

   However, since $|\phi_1\rangle$ and $|\phi_2\rangle$ form an orthonormal basis of the subspace of the Hilbert space where $\hat{\rho}$ has support, and since $|\psi_1\rangle$ is a unit vector that necessarily lives in this subspace,

   $$|\langle\psi_1|\psi_1\rangle|^2 = |\langle\psi_1|\phi_1\rangle|^2 + |\langle\psi_1|\phi_2\rangle|^2 = 1, \tag{9}$$

   which implies that

   $$\langle\psi_1|\psi_2\rangle = 0, \tag{10}$$

   as claimed. Equivalently, this means that if $\langle\psi_1|\psi_2\rangle \neq 0$, i.e., if $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal, then it must be that $\delta < 1$, and therefore that the states cannot be perfectly discriminated from each other.

## 8 Classical Mechanics

# Nollet's double cone

In Theoretical Mechanics, Nollet's double cone is a well-known mechanical paradox. It consists in a rigid body modeled as a double cone, which is apparently able to roll uphill along two inclined rails without sliding (see Fig. 1).

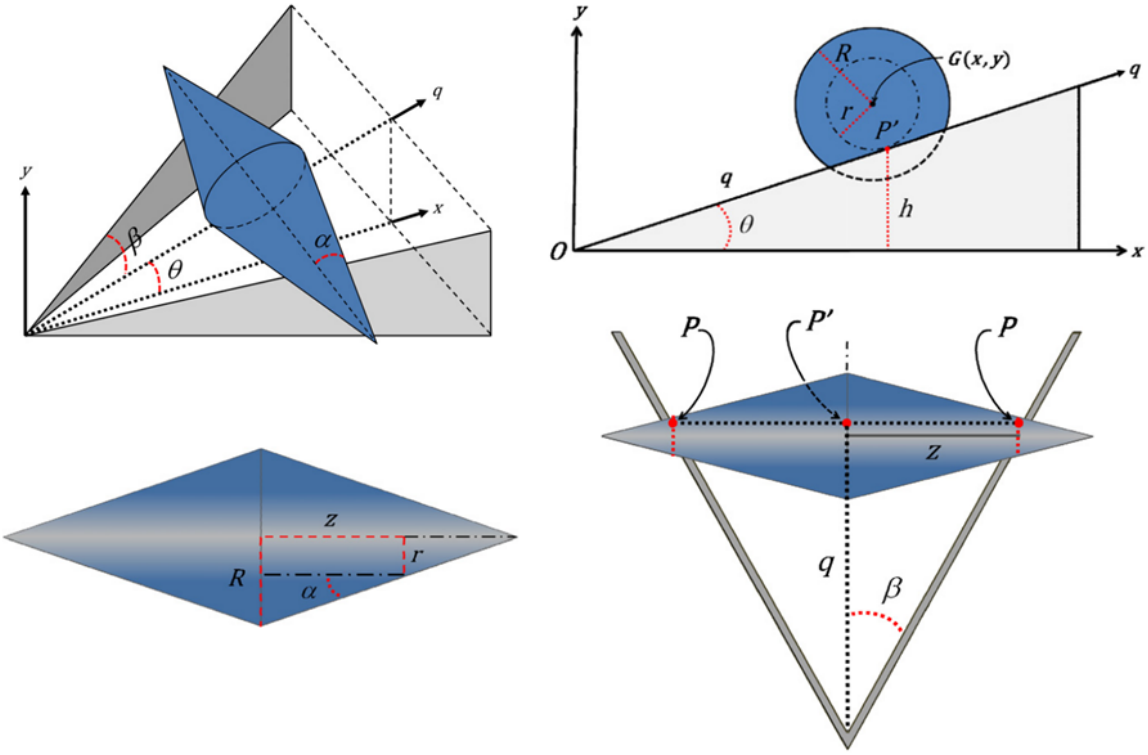

Figure 1: Nollet's double cone. Credit: Cortés & Cortés-Poza (2011).

In the different panels of Fig. 1, you can find the various quantities (angles, positions, radii, etc.) necessary to solve this problem. The double cone has a maximum radius $R$, and is placed on a pair of straight rails that form a "V" shape when seen from atop. The angle between the two "V"-shaped rails is $2\beta$, the inclination of the rails is $\theta$ (relative to the horizontal plane), and the cone's half-angle is $\alpha$. Other quantities relative to the cone's motion are also shown in Fig. 1, such as the displacement on the inclined rails, $q$; the contact points, $P$; the cone radius at the contact points, $r$; the distance of the contact point to the center of mass, $z$ (with $P'$ being the midpoint between $P$, the contact points); and the height of the double cone, $h$.

## QUESTIONS

1. **(2 points)** Compute the coordinates of the center of mass (CM), and show that its motion follows a straight-line trajectory, obtaining the equation for that line.
2. **(2 points)** Demonstrate that the center of mass is rolling uphill along the inclined rails when the condition

$$\tan\,\theta < \tan\,\alpha\,\tan\,\beta \tag{1}$$

is satisfied.

3. Obtain the equations of motion by using the Hamiltonian formalism. To this end:

   (a) **(1.5 points)** From the rolling without friction condition of the displacement, obtain $r$ as a function of $r = r(R, \phi, k)$, being $k = \tan\beta\,\tan\alpha$.

   (b) **(0.5 points)** Compute the center of mass coordinates $(x, y)$ by including the result from Question 3(a) into the result previously obtained in Question 1.

   (c) **(2 points)** Obtain the kinetic and potential energies, and hence, from them, the system's Lagrangian.

   (d) **(2 points)** Obtain the Hamiltonian and Hamilton's equations of motion.

## Solutions

From Fig. 1, we can obtain different geometrical and trigonometrical relations.

First of all, to obtain $\tan\theta < \tan\alpha\tan\beta$, some relations are obtained:

- Height of the contact point $P$:
$$h = q\sin\theta. \tag{2}$$
- On the other hand:
$$h = (R-r)\cos\theta. \tag{3}$$
- Distance of the contact point:
$$R - r = z\tan\alpha. \tag{4}$$
- Distance of the contact point to the center of mass:
$$z = q\tan\beta. \tag{5}$$

To demonstrate the relation, $h$, $r$, $R$, and $z$ must be eliminated, for example, from $h = q\sin\theta = (R-r)\cos\theta \Rightarrow R - r = q\tan\theta$. On the other hand, $R - r = z\tan\alpha$, thus: $q\tan\theta = z\tan\alpha$ and substituting $z = q\tan\beta \Rightarrow q\tan\theta = q\tan\beta\tan\alpha$:

$$\tan\theta = \tan\beta\tan\alpha. \tag{6}$$

1. **(2 points)** Compute the coordinates of the center of mass (CM), and show that its motion follows a straight-line trajectory, obtaining the equation for that line.

   Let us study the CM motion, located in Cartesian coordinates at $x$ and $y$. To this aim, we use Fig. 1. CM coordinates are:
$$x = q\cos\theta - r\sin\theta, \tag{7}$$
$$y = q\sin\theta + r\cos\theta. \tag{8}$$

   From Eqs. (4) and (5), $R - r = q\tan\beta\tan\alpha \Rightarrow r = R - q\tan\beta\tan\alpha$. By substituting in Eqs. (7) and (8):
$$x = q\cos\theta - (R - q\tan\beta\tan\alpha)\sin\theta, \tag{9}$$
$$y = q\sin\theta - (R - q\tan\beta\tan\alpha)\cos\theta. \tag{10}$$

   Rewriting Eqs. (9) and (10):
$$x + R\cos\theta = q(\cos\theta + \tan\beta\tan\alpha\sin\theta), \tag{11}$$
$$y - R\sin\theta = q(\sin\theta - \tan\beta\tan\alpha\cos\theta). \tag{12}$$

   Dividing Eqs. (11) and (12), it is obtained:
$$\frac{x + R\cos\theta}{y - R\sin\theta} = \frac{\tan\theta - \tan\beta\tan\alpha}{1 + \tan\theta\tan\beta\tan\alpha}. \tag{13}$$

   Rewriting Eq. (13):
$$y = \frac{\tan\theta - \tan\beta\tan\alpha}{1 + \tan\theta\tan\beta\tan\alpha}\,x + R\left(\frac{1 + \tan\beta\tan\alpha\tan\theta + \tan^2\theta - \tan\beta\tan\alpha\tan\theta}{1 + \tan\beta\tan\alpha\tan\theta}\right), \tag{14}$$

$$y = \frac{\tan\theta - \tan\beta\tan\alpha}{1+\tan\theta\tan\beta\tan\alpha}x + \frac{R}{1+\tan\beta\tan\alpha\tan\theta}, \tag{15}$$

where the property $1+\tan^2\theta = 1/\cos\theta$ has been used from simplifying Eq. (14) to Eq. (15). Note that Eq. (15) could be identify with a straight line given by $y = mx + n$, being $m$, the slope and $n$, the intercept:

$$m = \frac{\tan\theta - \tan\beta\tan\alpha}{1+\tan\theta\tan\beta\tan\alpha} \tag{16}$$

and

$$n = \frac{R}{1+\tan\beta\tan\alpha\tan\theta}. \tag{17}$$

2. **(2 points)** Demonstrate that the center of mass is rolling uphill along the inclined rails when the condition tan $\theta <$ tan $\alpha$ tan $\beta$ is satisfied.

   After taking into account the relations between different variable involved in the motion of the double cone and the demonstration of the straight line motion, the inequality would be achieved due to the motion of the CM is a straight line with slope $m < 0$ which is equivalent to $\tan\theta < \tan\beta\tan\alpha$.

3. Obtain the equations of motion by using the Hamiltonian formalism. To this end:

   (a) **(1.5 points)** From the rolling without friction condition of the displacement, obtain $r$ as a function of $r = r(R, \phi, k)$, being $k =$ tan $\beta$ tan $\alpha$.

   The Hamiltonian equations of motion are:

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{18}$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \tag{19}$$

   being $q$ a generalized coordinate, and $p$ a generalized momentum. As shown above, the motion is in 2D and a constraint is imposed by the rolling without friction, i.e.

$$dq = r d\phi, \tag{20}$$

   being $\phi$ the rotated angle. Now, it is called $k = \tan\beta\tan\alpha$. Lets used Eq. (20) by taking into account the relation: $R - r = kq \Rightarrow r = R - kq$ and substituting in Eq. (20):

$$dq = R - kq d\phi, \tag{21}$$

   and separating variables and integrating:

$$\frac{dq}{R-kq} = d\phi \Rightarrow \int_0^q \frac{dq}{R-kq} = \int_0^\phi d\phi \Rightarrow$$

$$-\frac{1}{k}\left[\log(R-kq) - \log R\right] = \phi \Rightarrow q = \frac{R}{k}\left[1-\exp(-k\phi)\right] \Rightarrow$$

$$r = R - kq = R\exp(-k\phi). \tag{22}$$

(b) **(0.5 points)** Compute the center of mass coordinates $(x, y)$ by including the result from Question 3(a) into the result previously obtained in Question 1.

Let us take into account Eq. (22) in the $(x, y)$ of the CM [Eqs. (14)]:

$$x = \frac{R}{k}\left[1 - \exp(-k\phi)\cos\theta - R\exp(-k\phi)\sin\theta\right], \tag{23}$$

$$y = \frac{R}{k}\left[1 - \exp(-k\phi)\sin\theta + R\exp(-k\phi)\cos\theta\right], \tag{24}$$

by reassembling

$$x = \frac{R}{k}\cos\theta - R\exp(-k\phi)\left(\frac{\cos\theta}{k} + \sin\theta\right), \tag{25}$$

$$y = \frac{R}{k}\sin\theta + R\exp(-k\phi)\left(\cos\theta - \frac{\sin\theta}{k}\right). \tag{26}$$

This equation provides the position of the CM.

(c) **(2 points)** Obtain the kinetic and potential energies, and hence, from them, the system's Lagrangian.

Now, we compute the kinetic and potential energies, the generalized momentum, construct the Hamiltonian and applied Hamilton's equations given by Eqs. (24):

- The motion is in 2-dimension x and y and we have a constraints, thus the system has only one degree of freedom.
- Generalized coordinate could be $q$ or $\phi$. In our case, $\phi$ is chosen.
- Translation kinetic energy: $T_{tras} = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2)$
  By deriving Eqs. (25) and (26) with respect to time, it achieved:

$$\dot{x}^2 + \dot{y}^2 = R^2\exp(-2k\phi)\dot{\phi}^2(1 + k^2). \tag{27}$$

  By substituting this result in the expression in translational kinetic energy:

$$T_{tras} = \frac{1}{2}MR^2\exp(-2k\phi)\dot{\phi}^2(1 + k^2). \tag{28}$$

- Rotational kinetic energy $T_{rot} = \frac{1}{2}I\dot{\phi}^2$.
  By taking into account the inertial moment $I = \frac{3MR^2}{10}$, the rotational energy is:

$$T_{rot} = \frac{1}{2}\frac{3MR^2}{10}\dot{\phi}^2. \tag{29}$$

- Gravitational potential energy: $V = Mgy$.
  By taking into account the coordinate of the CM:

$$V = Mg\left[\frac{R}{k}\sin\theta + R\exp(-k\phi)\left(\cos\theta - \frac{\sin\theta}{k}\right)\right]. \tag{30}$$

- Lagrangian: $L = T - V$
  Equations (28), (29), and (30):

$$L = \frac{1}{2}MR^2\dot{\phi}^2\left[\exp(-2k\phi)(1 + k^2) + \frac{3}{10}\right] - Mg\frac{R}{k}\sin\theta - MgR\exp(-k\phi)\left(\cos\theta - \frac{\sin\theta}{k}\right). \tag{31}$$

(d) **(2 points)** Obtain the Hamiltonian and Hamilton's equations of motion.

- Generalized momentum: $p_\phi = \frac{\partial L}{\partial \dot{\phi}}$.
  It yields:
$$p_\phi = MR^2\dot{\phi}\left[\exp(-2k\phi)(1+k^2) + \frac{3}{10}\right]. \tag{32}$$
- Hamiltonian: $H = T + V$ or $H = p_\phi\dot{\phi} - L$.
  We choose the first way:
$$H = \frac{p_\phi^2}{2MR^2\left[\exp(-2k\phi)(1+k^2) + \frac{3}{10}\right]} + Mg\frac{R}{k}\sin\theta + MgR\exp(-k\phi)\left(\cos\theta - \frac{\sin\theta}{k}\right). \tag{33}$$
- Hamilton's equations:
$$\dot{\phi} = \frac{\partial H}{\partial p_\phi}, \tag{34}$$
$$\dot{p_\phi} = -\frac{\partial H}{\partial \phi}. \tag{35}$$
  Deriving Eq. (35), it is obtained:
$$\dot{\phi} = \frac{p_\phi}{MR^2\left[\exp(-2k\phi)(1+k^2) + \frac{3}{10}\right]}, \tag{36}$$
$$\dot{p_\phi} = MgRk\exp(-k\phi)\left(\cos\theta - \frac{\sin\theta}{k}\right) - \frac{kp_\phi}{2MR^2\exp(-2k\phi)(1+k^2)}. \tag{37}$$